Instituto Superior Técnico

Digital Signal Processing (Processamento Digital de Sinais)

Test of 4/6/2016. Duration: 1.5 hours

Important notes:

- **Solve the problems in separate sheets.** You may solve several items of each problem in the same sheet.
- Identify all sheets with your student number and your first and last names.
- Present all answers in a clear, ordered and detailed manner.
- In calculations, keep at least three significant digits, and present all steps, with a brief explanation of each one.
- **Justify all answers and all calculation steps.**

Problem 1

Consider a Gaussian random variable x with mean $2A$ and variance $15A^2$. Assume that we have available a single observation of the random variable.

a) Find the expression of the maximum-likelihood estimator of A.

b) Is the estimator that you obtained unbiased? If you were not able to solve item a), use $\hat{A} = x/4$ in this item.

c) Find the Cramér-Rao bound (CRB) for the estimation of A. Don't forget to check whether there is a CRB for this estimation problem.

Problem 2

We wish to approximate a set of observations, $\{x_n, n = 0, \cdots, N-1\}$ with the model $x_n = a^n$, using the least-squares criterion.

a) Find the expression of the least-squares estimator of a for $N = 2$.

b) Is it possible to obtain an explicit expression of the least-squares estimator of a, valid for any value of N?

c) Taking logarithms of both sides of the equation of the model given above, we get the equation $\log x_n = n \log a$. Making $y_n = \log x_n$ and $b = \log a$, we get the model equation $y_n = nb$. Find the expression of the least-squares estimator of b for this model. The expression should be valid for any value of N.

d) Designating by $\hat{b}$ the least-squares estimator of b mentioned in item c), we can obtain an estimator of a, for any value of N, as $\hat{a} = e^{\hat{b}}$. Discuss the validity of this procedure as a method to obtain the least-squares estimator of a.You may be able to answer this item even if you haven't answered item c).

Problem 3

Consider a random variable x with the probability density $p(x) = a\, e^{-ax} u(x)$, with $a > 0$. The prior probability density of a is $p(a) = 9a\, e^{-3a} u(a)$. We have available three independent observations of x, $\{x_1, x_2, x_3\} = \{3, 5, 4\}$. Find the maximum-a-posteriori estimate of a, based on these observations.

Problem 4

Read both items of this problem before starting to solve any of them, to better understand what is requested in each one.

Consider a Bayesian estimation problem with a single parameter $\theta \in \mathbb{R}$ and a single observation $x \in \mathbb{R}$.

a) Explain what is the quantity that is optimized when one finds the minimum-mean-square-error estimator of θ. Explain whether that quantity is minimized or maximized, and why it makes sense to optimize it. Include, in your explanation, the mathematical expression of the mentioned quantity. If that expression involves any expected values, write them explicitly in terms of integrals.

b) Give the general expression of the MMSE estimator, i.e., of the solution of the optimization mentioned in item a). If the expression involves any expected values, write them explicitly in terms of integrals.

Problem 5 (*2 marks*)

This problem is intended to distinguish the students that deal best with the topics studied in our course. In this problem, you must provide a very detailed and very carefully justified response.

Consider a stationary random process x with autocorrelation function $r_x(n) = 2^{-|n|}$. Assume that the values of x_0 and x_2 are known. Find the MMSE linear estimator of x_1, i.e., the estimator of x_1 with the lowest mean-square error among all estimators of the form $\hat{x}_1 = ax_0 + bx_2$.