**Instituto Superior Técnico**

**Digital Signal Processing (Processamento Digital de Sinais)**

**Exam of 27/6/2016. Duration: 3 hours**

***Important notes:***

- **Solve the problems in separate sheets.** You may solve several items of each problem in the same sheet.
- Identify all sheets with your student number and your first and last names.
- Present all answers in a clear, ordered and detailed manner.
- In calculations, keep at least three significant digits, and present all steps, with a brief explanation of each one. Simplify the final results of each item, whenever possible.
- **Justify all answers and all calculation steps.**
- The simplicity of the methods that you use to solve the various problems will be taken into account in the grading.

**Problem 1**

A continuous-time signal $x_c$ is sampled with a sampling period of $0.01\pi$, resulting in the discrete-time signal $x_d$.

a) Find expressions of $x_d$ for the following cases. The expressions should be as simple as possible.

   i) (*1.2 marks*) $x_c(t) = \cos(10\,t)$.

   ii) (*0.6 marks*) $x_c(t) = u(t + 0.02) - u(t - 0.02)$.

b) (*1.5 marks*) The signal $x_d$ is passed through a *real* discrete-time LTI filter whose frequency response is $j\omega + 3$ for $0 \le \omega < \pi$. The output of the filter goes through an ideal reconstructor for the above-mentioned sampling period, resulting in the signal $y_c$. For a certain class of signals $x_c$, the system with input $x_c$ and output $y_c$ is equivalent to a continuous-time LTI filter. Indicate what is that class of signals, and find the frequency response of the mentioned equivalent continuous-time LTI filter.

**Problem 2**

Consider the signals $x(n) = u(n) - u(n-20)$ and $y(n) = 2^{n-3}[u(n-3) - u(n-5)]$.

a) (*1.6 marks*) Find their linear (i.e., non-circular) convolution.

b) (*0.5 marks*) Find their length-20 circular convolution.

c) (*1.6 marks*) Find the length-5 DFT of $y$. Give the numerical values of the DFT samples, in your response.

**Problem 3** (*2 marks*)

A stationary, discrete-time random process with autocorrelation function $2^{-|n|}$ is placed at the input of an LTI system with impulse response $3^{-n}u(n)$. Find the power spectrum of the system's output.

**Problem 4**

Consider a random variable $x$ with probability density

$$p(x;a) = \begin{cases} a x^{a-1} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise,} \end{cases} \qquad \text{where } a > 0.$$

Assume that we have $N$ independent observations of $x$.

a) (*1.9 marks*) Find the maximum-likelihood estimator of $a$.

b) (*1.4 marks*) Find the Cramér-Rao bound (CRB) for the estimation of $a$. You may assume that the bound exists.

**Problem 5** (*1.9 marks*)

A certain system has two internal variables, $x_1$ and $x_2$. We observe three external variables, $y_1$, $y_2$ and $y_3$, that depend on the internal ones, and we have a model of the relationship between internal and external variables:

$$\begin{cases} y_1 = x_1 \\ y_2 = x_1 + x_2 \\ y_3 = x_1 - x_2\,. \end{cases}$$

In a certain situation, we observe the values $y_1 = 1$, $y_2 = 4$, and $y_3 = 0$. Find the least-squares estimates of $x_1$ and $x_2$ for this situation.

**Problem 6** (*1.9 marks*)

Consider a continuous random variable $x$ with uniform distribution in the interval $[0, \theta]$. The parameter $\theta$ has the prior probability density

$$p(\theta) = \begin{cases} \frac{a}{\theta^2} & \text{if } \theta \geq a \\ 0 & \text{otherwise,} \end{cases} \qquad \text{where } a \text{ is a positive constant.}$$

Assume that we have available a single observation of $x$. Find the maximum-a-posteriori estimator of $\theta$.

**Problem 7** (*1.9 marks*)

Explain what is a maximum-a-posteriori estimator. Also explain why this is a reasonable estimator; the latter explanation should be based on the consideration of the Bayes risk. Be precise in your response, using mathematical expressions where appropriate.

**Problem 8** (*2 marks*)

*This problem is intended to distinguish the students that deal best with the topics studied in our course. In this problem, you must provide a very detailed and very carefully justified response.*

Consider two jointly distributed random variables, $x$ and $y$, whose distribution depends on a scalar parameter $\theta$. Intuitively, on average, the information on $\theta$ contained in the pair $(x, y)$ should not be lower than the information contained in $x$ alone. In this problem, you are requested to prove that that is indeed true. More specifically: show that the Fisher information on $\theta$ based on the distribution of the pair $(x, y)$ is greater than or equal to the Fisher information on $\theta$ based on the distribution of $x$. You may make any assumptions that are needed for the proof.

Suggestion: Express the joint probability density of $x$ and $y$ in terms of the conditional probability density of one of the variables, given the other.