Example problems on Support Vector Machines

Problem 1

Notes: This problem is longer than a normal exam problem would be, but is useful for letting the students deal with several aspects of SVMs. The problem requires the use of square-lined paper. If the exam requires the use of square-lined paper, that paper will be supplied to the students at the exam.

Consider a classification problem in $\mathbb{R}^2$, with two classes, in which the training set is given by:

x_1	x_2	Class
–2	–2	A
-2	-1	A
1	2	A
2	1	A
-2	2	B
0	2	B
0	-1	B
2	-1	B

Consider the nonlinear mapping from input space to a two-dimensional feature space, given by

$$(x_1, x_2) \to (x_1^2, x_1 x_2)$$

a) Plot, on square-lined paper, the training patterns in input space, and label them according to the class they belong to. State whether the patterns from the two classes are linearly separable in this space.

b) Plot, on square-lined paper, separately from the graph made in step a), the training patterns in feature space, and label them according to the class they belong to.

c) Find the widest-margin classifier in feature space. More specifically, find the equations of the classification boundary and of the two margin boundaries. Plot these three boundaries on the same graph that was used in step b). Also indicate which are the support vectors in feature space. Note that the boundaries and the support vectors are easy to find by inspection.

d) Find which vectors, in input space, correspond to the support vectors found in step c).

e) Plot the classification boundary in input space, on the same graph that was used in step a).

Note that:

- Equations of the boundary in input space, in the forms $x_1 = f(x_2)$ and $x_2 = g(x_1)$, are easy to derive from the classification boundary equation found in step c).
- The boundary is a hyperbola whose asymptotes are easy to find from the first of those equations, by assuming that x_2 is large in absolute value.
- Using the second of those equations, it is easy to find points of the hyperbola for $x_1 = \pm 1/2$ and for small integer values of x_1 (both positive and negative).

f) Plot, on the same graph, the boundaries of the classification margin zone in input space. Shade the area between these two boundaries, to better visualize the classification margin zone in input space.

g) Using the graph completed in step f), check that the training patterns from different classes fall on different sides of the classification boundary, that the support vectors found in step d) fall on the margin boundaries, and that no training patterns fall within the classification margin zone.

h) Write the inequality that you would use to classify new input patterns with the SVM classifier developed in steps a) to g). Choose an inequality that is as simple as possible.

Problem 2

Consider a support vector machine whose input space is $\mathbb{R}^2$, and in which the inner products are computed by means of the kernel

$$k(\mathbf{x},\mathbf{y}) = (\mathbf{x}\cdot\mathbf{y}+1)^2 - 1$$

(bold letters represent vectors in $\mathbb{R}^2$ and $\mathbf{x}\cdot\mathbf{y}$ denotes the ordinary inner product in $\mathbb{R}^2$).

Show that the mapping to feature space that is implicitly defined by this kernel is the mapping to $\mathbb{R}^5$ given by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightarrow \phi(\mathbf{x}) = \begin{bmatrix} x_1^2 \\ x_2^2 \\ \sqrt{2}\,x_1x_2 \\ \sqrt{2}\,x_1 \\ \sqrt{2}\,x_2 \end{bmatrix}.$$