From diversity and denoising to phase imaging

Gonçalo Valadão and José Bioucas-Dias
Instituto de Telecomunicações, Av. Rovisco Pais,1, 1049-001 Lisboa, Portugal
Phone: +35121841846{7,6}, Fax: +351218418472, email:{gvaladao,bioucas}@lx.it.pt

***Abstract*— Many imaging techniques, *e.g.*, magnetic resonance imaging (MRI), yield phase images. In these, each pixel retrieves the phase up to a modulo-2π rad ambiguity, *i.e.*, the phase wrapped around the principal interval $[-\pi\ \ \pi($. Phase unwrapping (PU) is, then, a crucial operation to obtain absolute phase, which is what embodies physical information. If the phase difference between neighbor pixels is less than π rad, then, phase unwrapping can be obtained unambiguously. This, however, is not always the case. *E.g.*, in MRI, where absolute phase can be proportional to temperature, we often face neighbor phase differences much larger than π rad. The PU problem is even more challenging for noisy images. This paper proposes a diversity approach, which consists of using two (or more) images of the same scene acquired with different frequencies. Diversity grants an enlargement of the ambiguity interval $[-\pi\ \ \pi($, thus, allowing to unwrap images with high phase rates. Furthermore, this paper presents a multi-resolution technique to make denoising. We formulate the problem with a *maximum a posteriori* - Markov random field (MAP-MRF) rationale, and apply energy minimization techniques based on graph cuts. We illustrate the performance of the algorithm by showing experimental results, and argue that it is, as far as we know, state-of-the art competitive.**

I. INTRODUCTION

There are nowadays many applications based on phase images. Namely, interferometric synthetic aperture radar (InSAR) is used to the generation of digital elevation models, and magnetic resonance imaging (MRI) in, *e.g.*, angiography. In all of these imaging systems, the acquisition sensors read only the cosine and the sine components of the absolute phase; that is, we have access only to the phase modulo-2π, the so-called interferogram. Besides the sinusoidal nonlinearity, the observed data is corrupted by noise. Due to these degradation mechanisms, phase unwrapping is known to be a very difficult problem. In fact, if the magnitude of phase variation between neighbor pixels is larger than π, *i.e.*, the so-called Itoh condition [1] is violated, then the inference of the 2π multiples is an ill-posed problem. These violations may be due to undersampling, discontinuities, or noise.

Frequency diversity based PU algorithms are scarce. We are aware only of the ones proposed in [2], [3], and [4]. Regarding the first one it presents interesting but error prone algorithms. The second one is a multidimensional version of the minimum L^2 norm type of PU algorithm [5], with relaxation to the continuum. The weaknesses of this approach are long-familiar in particular the oversmoothing of high phase rate slopes. and discontinuities. Concerning the third, it consists of an algorithm whose goal is to approximate the true surface by means of local planes. The proposed approach requires a simulated annealing computation which is a (nowadays) too much slow optimization technique.

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A. Contributions

The main contribution of this paper is to present an algorithm that accomplishes both phase unwrapping and denoising.

Our approach is Bayesian. We adopt an observation model that is 2π-periodic, and discontinuity preserving MRF priors for the absolute phase. A MAP criterion infers the phase by exploiting graph-cuts based energy minimization techniques. The algorithm has two main steps:

1) Phase unwrapping: we input two (or more) different frequency interferograms (of the same scene), which provides an extension of the $[-\pi, \pi)$ ambiguity interval and, consequently, an increasing of the phase rates that still allow unwrapping to be a well-posed problem. This frequency *diversity* technique is put forward through a graph-cuts algorithm [6] that minimizes a MRF composed of a sinusoidal data term plus a non-isotropic total variation (TV) prior [6].
2) Denoising: we achieve denoising by an iterative *multi-precision* MAP-MRF energy minimization graph-cuts algorithm. As in the previous step, (Phase unwrapping), the data term is sinusoidal, while a discontinuity preserving denoising prior is considered [7], [8].

II. PROPOSED FORMULATION

A. Posterior density

We consider, as in, *e.g.*, [9], the observation data model to be given by (1)

$$z = e^{jF\phi} + n, \tag{1}$$

where F has the meaning of frequency, ϕ is the absolute phase, and n a zero-mean, circular, Gaussian noise. The log-likelihood of ϕ, given the observed $\psi = \mathrm{angle(z)}$, is basically (see [9]) given by (2)

$$f(\phi|\psi) = -\cos(\psi - F\phi). \tag{2}$$

Considering also that the prior is a MRF, then the logarithm of the posterior density is given by (3)

$$E(\phi) \equiv \underbrace{\sum_{i\in\mathcal{V}} -\cos(\psi_i - F\phi_i)}_{\text{Data fidelity term}} + \underbrace{\mu \sum_{(i,j)\in\mathcal{E}} V(\phi_i, \phi_j)}_{\text{Prior term}}, \tag{3}$$

where $\phi = (\phi_1, \phi_2, \ldots, \phi_{|\mathcal{V}|})$, V is a, to be specified, potential and finally, μ is the prior parameter that sets the relative weight between the data fidelity and the prior terms.

B. Diversity

In this paper we consider frequency diversity which is particularly used in various areas, such as, *e.g.*, MRI, echographic Doppler, weather radar, and InSAR. Namely, we deal with two (for the sake of simplicity) frequencies $F_1 = p/q$, $F_2 = r/s$ where $\{p, q, r, s\} \in \mathbb{N}$. Assuming that observations (1) are independent for each frequency, the log-likelihood is given by (4)

$$f(\phi|\psi_1, \psi_2) = -\cos(\psi_1 - F_1\phi) - \cos(\psi_2 - F_2\phi). \quad (4)$$

We have already alluded (Section I-A) to the advantage that frequency diversity gives in extending the $[-\pi \ \pi[$ ambiguity interval. Stating it more clearly, it is easy to show that the sum of two cosine functions, having as in (4) different frequencies $F_1 = p/q$ and $F_2 = r/s$, where $\{p, q\}, \{p, r\}$, $\{q, s\}$, and $\{r, s\}$ are coprime integers[1], results in a third periodic function whose period is $q \times s$; as the initial functions do have periods of respectively q and s, we conclude that the period is, in general, extended and so the ambiguity reduced. This is the "beat production", long known in wave physics. It is a well known behavior, *e.g.*, from wave phenomena, that the greater the beat period extension, the smaller the difference between global and local maxima. Furthermore, it is also well known that beat period extension brings noise amplification. This trade-off should then be taken into account.

C. Phase unwrapping with diversity

Replacing the data fidelity term in (3) with that given by (4), *i.e.*, considering diversity, we obtain

$$\begin{aligned} E(\phi) \ \equiv \ & \sum_{i\in\nu} -\cos(\psi_{1_i} - F_1\phi_i) - \cos(\psi_{2_i} - F_2\phi_i) \\ + \ & \mu \sum_{(i,j)\in\mathcal{E}} V(\phi_i, \phi_j). \end{aligned} \quad (5)$$

In this section we solve only the phase unwrapping problem. So, by admitting a noiseless environment, we may consider that the unwrapped (true) phase ϕ is given by:

$$F_1\phi = \psi_1 + 2k_1\pi, \quad (6)$$

and

$$F_2\phi = \psi_2 + 2k_2\pi, \quad (7)$$

for the two independent observations with frequencies F_1 and F_2, respectively. For the sake of simplicity we can deal with (6) only, and (5) turns into:

$$\begin{aligned} E(k) \ \equiv \ & \sum_{i\in\mathcal{V}} -\cos\left(\psi_{2_i} - \frac{F_2}{F_1}(\psi_{1_i} - 2k_{1_i}\pi)\right) \\ + \ & \mu \sum_{(i,j)\in\mathcal{E}} V\left(\psi_{1_i} - 2k_{1_i}, \psi_{1_j} - 2k_{1_j}\right), \end{aligned} \quad (8)$$

with a correspondingly combinatorial optimization (minimization) to be done on variables k_{1_i}. We take $V\left(\psi_{1_i} - 2k_{1_i}, \psi_{1_j} - 2k_{1_j}\right) = |k_i - k_j|$ the, so-called, non-isotropic total variation (TV); in spite of such a potential being convex, which confers some optimization "easiness", it still has some discontinuity preservability properties[2].

We are aware of only three integer optimization algorithms, that are able to provide a global minimum for a posterior energy like (8), which is composed by a non-convex data fidelity term and a convex prior potential. Herein we refer to [6], as it deals with our non-isotropic TV prior. As long as the energy is a levelable function, i.e., a function that admits a decomposition as a sum on levels, of functions of its variables level-set indicatrices at current level (see [6]), it is easy to build a source-sink graph such that its min-cut gives the sought global minimizer. For the sake of simplicity we do not describe the above mentioned energy decomposition on level-set dependent functions, as we will not describe the graph construction; we just mention that graph min-cuts based algorithms have been proven very popular in computer vision, as there are plenty of low-order polynomial complexity algorithms to compute them.

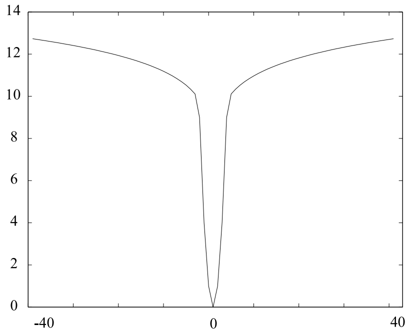

Fig. 1. Plot of a half-quadratic potential

D. Denoising with multi-precision

For denoising we take a half quadratic potential type like the one plotted in Fig.1. This potential is quadratic in an origin neighborhood of radius π in order to model Gaussian noise, and with a flat trend elsewhere to preserve discontinuities [10]. We choose the radius of π because we expect to get (most of the) noise wrapped into the interval $[-\pi, \pi($ after the previous phase unwrapping step.

For the sake of clarity we refer back to the posterior density expression (3), which writes energy as $E \equiv E(\phi)$. Our goal is to compute $\phi^* = \arg\min [E(\phi)]$. We note that the objective function, $E(\phi)$, is non-convex (both in the data fidelity term and in the prior term), which makes this optimization problem very difficult. To circumvent this problem we discretize the domain of E, using a discretization interval Δ. In doing this, we convert the minimization in $\mathbb{R}^{|\mathcal{V}|}$, where $\mathcal{V}$ denotes the set of pixels, into a combinatorial problem that may be solved efficiently by computing flows on appropriate graphs. Furthermore, we choose to make the denoising optimization using one frequency data only. Otherwise the objective function gets a huge amount of local minima and, as a result, the problem is

[1] Two integer numbers are said to be coprime if their greatest common divisor is the unity.

[2] In a sense, this can be interpreted as the most nonconvex of the convex functions

intractable. In doing this we are deciding for a sub-optimal solution.

We adopt a strategy in which the minimum of E is searched for in a sequence of increasing precisions. This way we both avoid getting stuck in local minima (which would be probable, had we started with high precision), and we probably get close to optimization in $\mathbb{R}^{|\mathcal{V}|}$. To this end, let us define $i \in \mathcal{V}$, $\delta_i \in \{0,1\}$, and the sets

$$\begin{aligned} M^U(\phi', \Delta) &\equiv \left\{\phi \in \mathbb{R}^{|\mathcal{V}|} : \phi_i = \phi'_i + \delta_i \Delta\right\} \\ M^D(\phi', \Delta) &\equiv \left\{\phi \in \mathbb{R}^{|\mathcal{V}|} : \phi_i = \phi'_i - \delta_i \Delta\right\}, \end{aligned}$$

where $\Delta \in \mathbb{R}$.

Algorithm 1 shows the pseudo-code for our optimization scheme.

Algorithm 1 Multi-precision denoising

Initialization: $\phi = \psi$ {Interferogram}, successup = false, successdown = false
1: **for** $\Delta = 2\pi \times \left\{2^0, 2^{-1}, \ldots, 2^{-N}\right\}$ **do**
2:   **while** (successup = false OR successdown = false) **do**
3:     **if** successup = false **then**
4:       $\hat{\phi} = \arg\min_{\hat{\phi} \in M^U(\phi,\Delta)} \tilde{E}(\hat{\phi})$
5:       **if** $E(\hat{\phi}) < E(\phi)$ **then**
6:         $\phi = \hat{\phi}$
7:       **else**
8:         successup = true
9:       **end if**
10:    **end if**
11:    **if** successdown = false **then**
12:      $\hat{\phi} = \arg\min_{\hat{\phi} \in M^D(\phi,\Delta)} \tilde{E}(\hat{\phi})$
13:      **if** $E(\hat{\phi}) < E(\phi)$ **then**
14:        $\phi = \hat{\phi}$
15:      **else**
16:        successdown = true
17:      **end if**
18:    **end if**
19: **end while**
20: **end for**

Our algorithm engages on a greedy succession of up and down binary optimizations. The precision of the minimization, Δ, starts with the value 2π and ends with the value $2\pi/(2^N)$ where N is a depth of precision. We point out that even if all the computations could have been done with the highest Δ resolution level from the very beginning, choosing this multi-resolution schedule increases dramatically (a logarithmic improvement) the algorithm speed.

To solve the binary optimizations shown in lines 4 and 12 of Algorithm 1, we use the graph-cuts technique presented in [11]. We further add that the $\tilde{E}$ is a majorizer, on the prior terms, of E. So we apply a majorize-minimize (MM) [12] technique such as the one applied in [7]. For details see, *e.g.*, [7], [11]. We stress that we do not have any guarantees of reaching a global minimum with Algorithm 1. This is so because, with generality, we are dealing with both non-convex data fidelity terms and prior terms. However, results in a series of experiments on simulated and real data have been systematically state-of-the art.

III. PROPOSED ALGORITHM

The previous sections culminate in our phase imaging algorithm. It consists of a phase unwrapping stage and then denoising. Algorithm 2 shows a simple two lines high level pseudo-code of our phase imaging algorithm.

Algorithm 2 Phase imaging algorithm

Do phase unwrapping with diversity
Do denoising with multi-resolution

In the next section we show some relevant experimental results.

IV. EXPERIMENTAL RESULTS

In this section, we briefly illustrate the performance of our algorithm on two representative problems for wich phase unwrapping is a hard problem due to high phase rates of the unwrapped images.

Fig. 2 (a) displays an image which is given by a Gaussian having maximum height of 50π rad height. Figs. 2 (b) and (c) show the corresponding wrapped images acquired with frequencies $F_1 = 1/1$ and $F_2 = 1/5$ respectively and having signal to noise ratio SNR $= 4$ dB. Fig. 2 (d) displays an image of the unwrapped Gaussian, and Fig. 2 (e) a 3-D rendering. Fig. 2 (f) shows a 3-D rendering after the denoising. It is clear that the algorithm made a perfect phase unwrapping (up to a no-meaning additive constant) for which the diversity information was essential. Concerning the denoising step, the performance is characterized by a improvement ISNR $= 0.0187$ dB. This, slight denoising is reflected in Figs. 2 (h) and (i), which show the histograms (the axis are in rad) corresponding to the surfaces rendered in Figs. 2 (e) and (f), respectively. It is noticeable that the denoising erases the secondary modes in the histogram.

Fig. 3 is similar to the Fig. 2 but the starting image is sheared quadratic ramp having maximum height of 225 rad. The frequencies of acquisition are $F_1 = 1/1$ and $F_2 = 1/11$ respectively and have SNR $= 7$ dB. Again the algorithm made a perfect phase unwrapping for which the diversity information was essential. Concerning the denoising, it has ISNR $= 5.4792$ dB. It is noticeable that the denoising erases the secondary modes in the first histogram.

Still referring to the histograms, the ones corresponding to the noisy images show, in general, a multi modal shape. Besides the central mode around zero, there are some modes around multiples of -2π and 2π. Those correspond to "spikes" as a result of the data observation model. After denoising they do disappear.

V. CONCLUDING REMARKS

We have proposed a phase imaging algorithm based on phase unwrapping with diversity, and denoising with multi-precision. Our approach is a MAP-MRF one. We have chosen

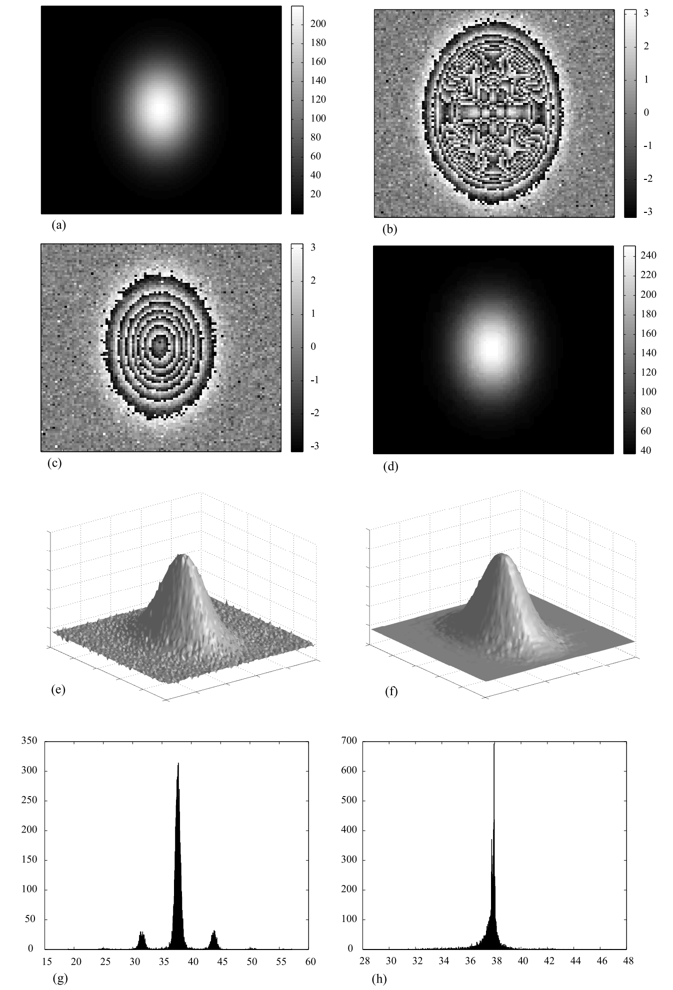

Fig. 2. (a) Original Gaussian ramp phase image. (b) Image in (a) wrapped with a relative frequency of 1. (c) Image in (a) wrapped with a relative frequency of $1/5$. (d) Unwrapped image from the previous wrapped images shown in (b) and (c). (e) 3-D rendering of the image in (d). (f) 3-D rendering of the image in (d) after the denoising step. (g) Histogram corresponding to the surface rendered in (e). (h) Histogram corresponding to the surface rendered in (f).

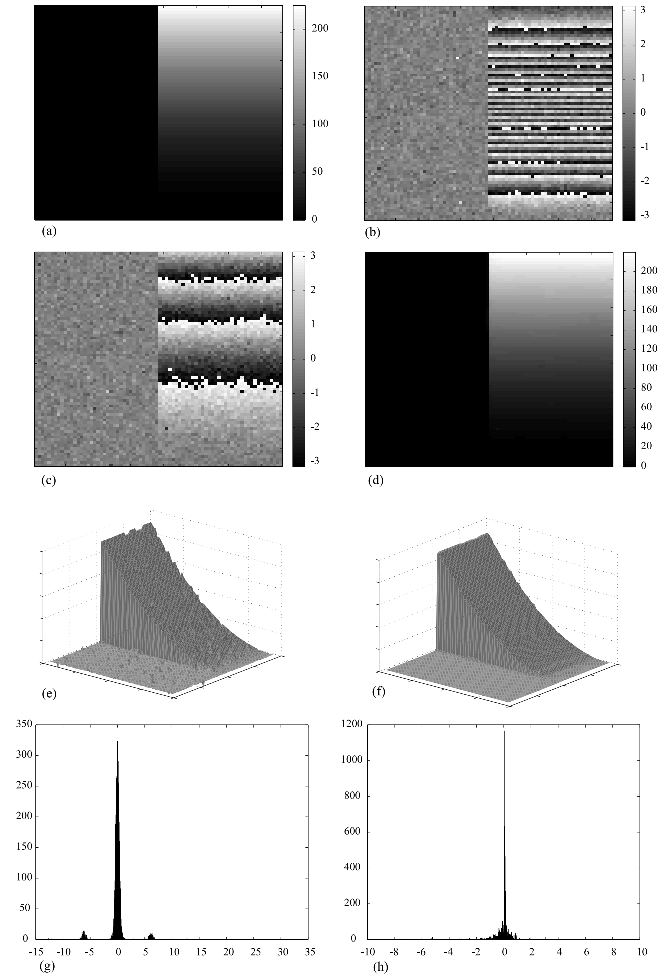

Fig. 3. (a) Original sheared quadratic ramp phase image. (b) Image in (a) wrapped with a relative frequency of 1. (c) Image in (a) wrapped with a relative frequency of $1/11$. (d) Unwrapped image from the previous wrapped images shown in (b) and (c). (e) 3-D rendering of the image in (d). (f) 3-D rendering of the image in (d) after the denoising step. (g) Histogram corresponding to the surface rendered in (e). (h) Histogram corresponding to the surface rendered in (f).

both non-convex data fidelity and prior potential terms, in the MRF, so there is no hope to find the global minimum efficiently. Thus, we propose a sub-optimal minimization based on graph cuts. Our approach inherits much of the PUMA algorithm [7], however, we do extend it by taking into account a data fidelity term and a denoising operation. The results are encouraging; to our knowledge they are state-of-the-art.

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