X-Ray computed tomography

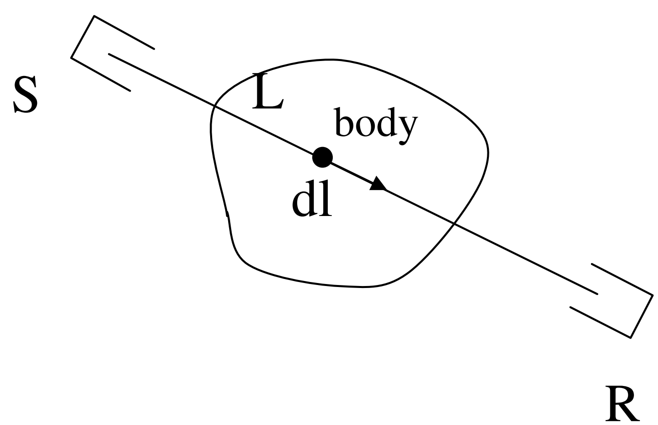
- Radon Transform
- Fourier Slice Theorem
- Backprojection Operator
- Filtered Backprojection (FBP) Algorithm
- Implementation Issues
- Total Variation Reconstruction

X-Ray tomography

- "Tomography" comes from Greek and means cross-sectional representations of a 3D object

- X-ray tomography, introduced by Hounsfield in 1971, is usually called *computed tomography* (CT)

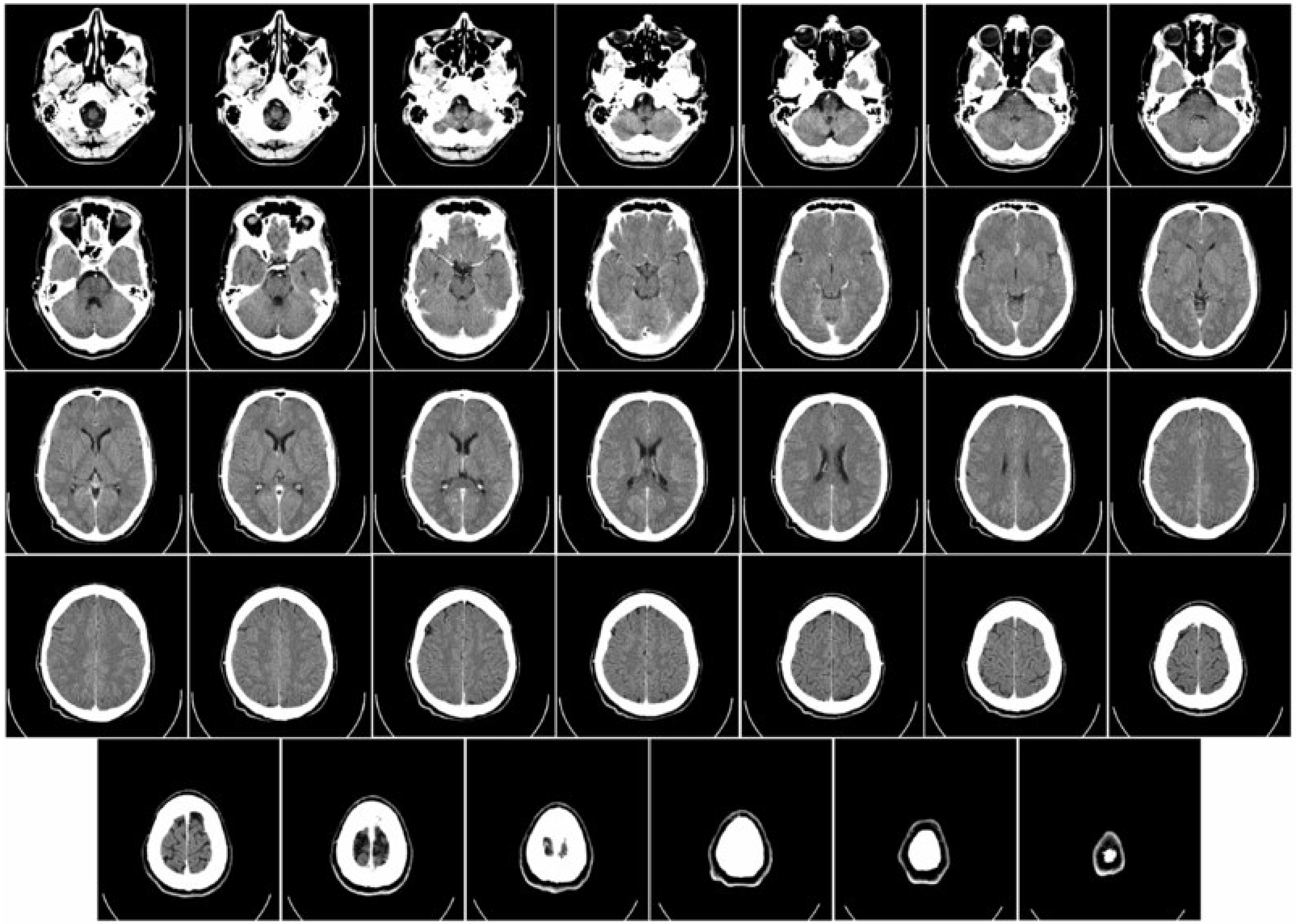
- CT produces images of human anatonomy with a resolution of about 1mm and an attenuation coefficient attenuation of about 1%



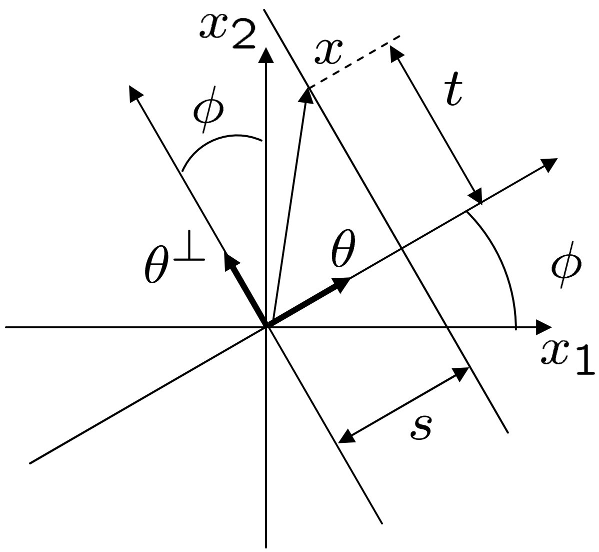
$$I = I_0 \exp\left(-\int_L f(\mathbf{x}(l))dl\right)$$

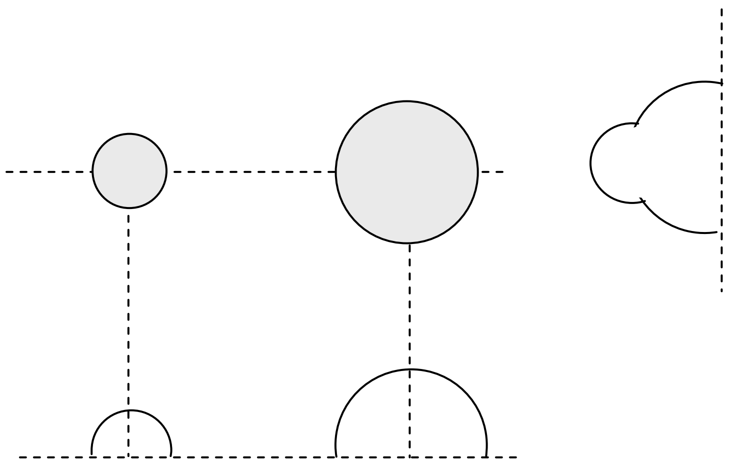
$$\log I = \log I_0 - \int_L f(\mathbf{x}(l))dl$$

CT images (from wikipedia)



X-Ray tomography





Radon transform $$(Rf)(s,\theta) = \int f(\underbrace{s\theta + t\theta^\perp}_{x})dt$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}$$

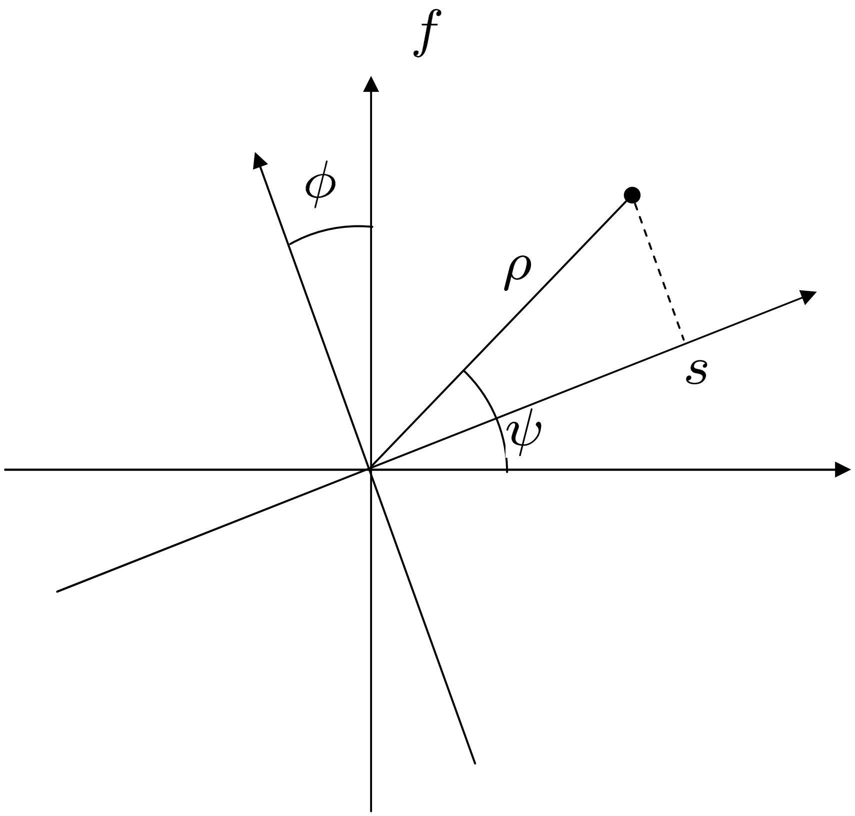
$$\theta = (\cos\phi, \sin\phi)$$

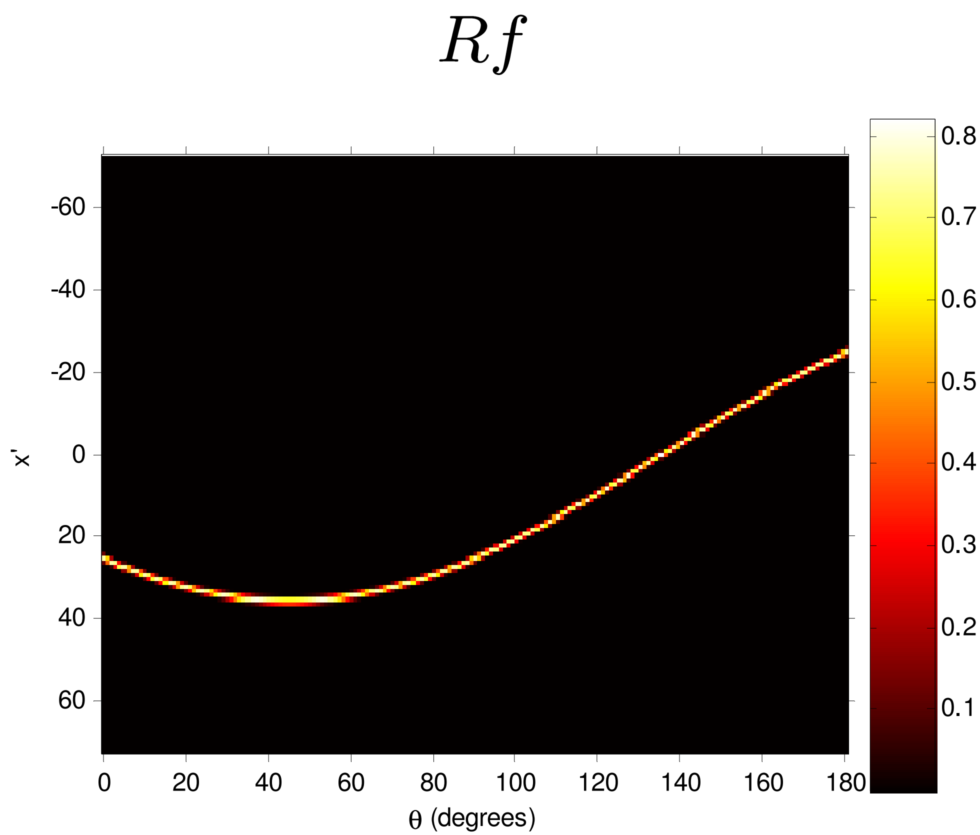
$$\theta^\perp = (-\sin\phi, \cos\phi)$$

$$-\infty < s < \infty, \; -\pi < \phi \leq \pi$$

$$(Rf)(-s,-\theta) = (Rf)(s,\theta)$$

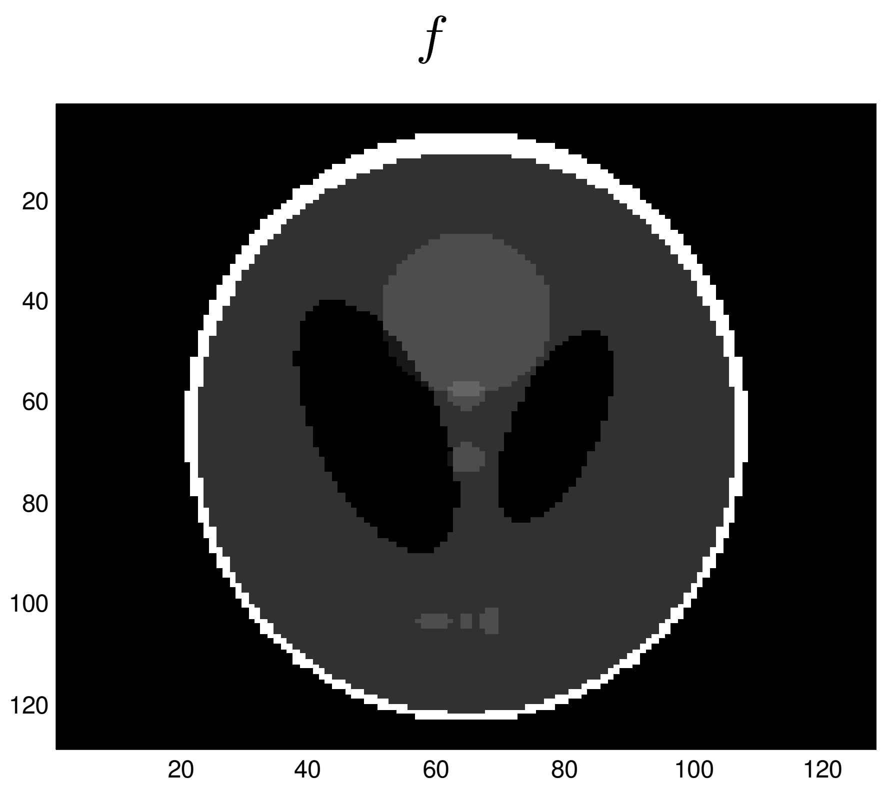
Example of Radon transform: sinogram

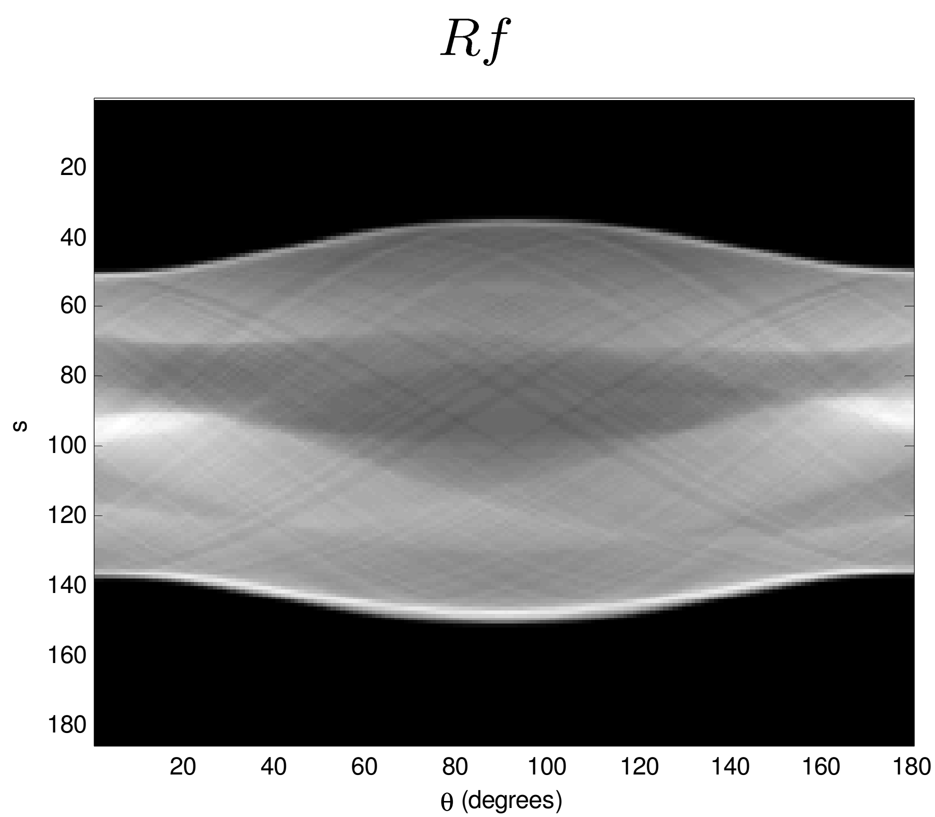




$$s = \rho \cos(\psi - \phi)$$

Example of Radon transform: sinogram





Fourier slice theorem

$$(Rf)(s,\theta) = \int f(s\theta + t\theta^{\perp})dt = g(s,\theta)$$

$$\tilde{g}(\omega,\theta) = \int e^{-j\omega s} g(s,\theta)\, ds$$

$$= \int e^{-j\omega s} \left(\int f(\underbrace{s\theta + t\theta^{\perp}}_{x})dt \right) ds$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} \qquad s = \theta \cdot x$$

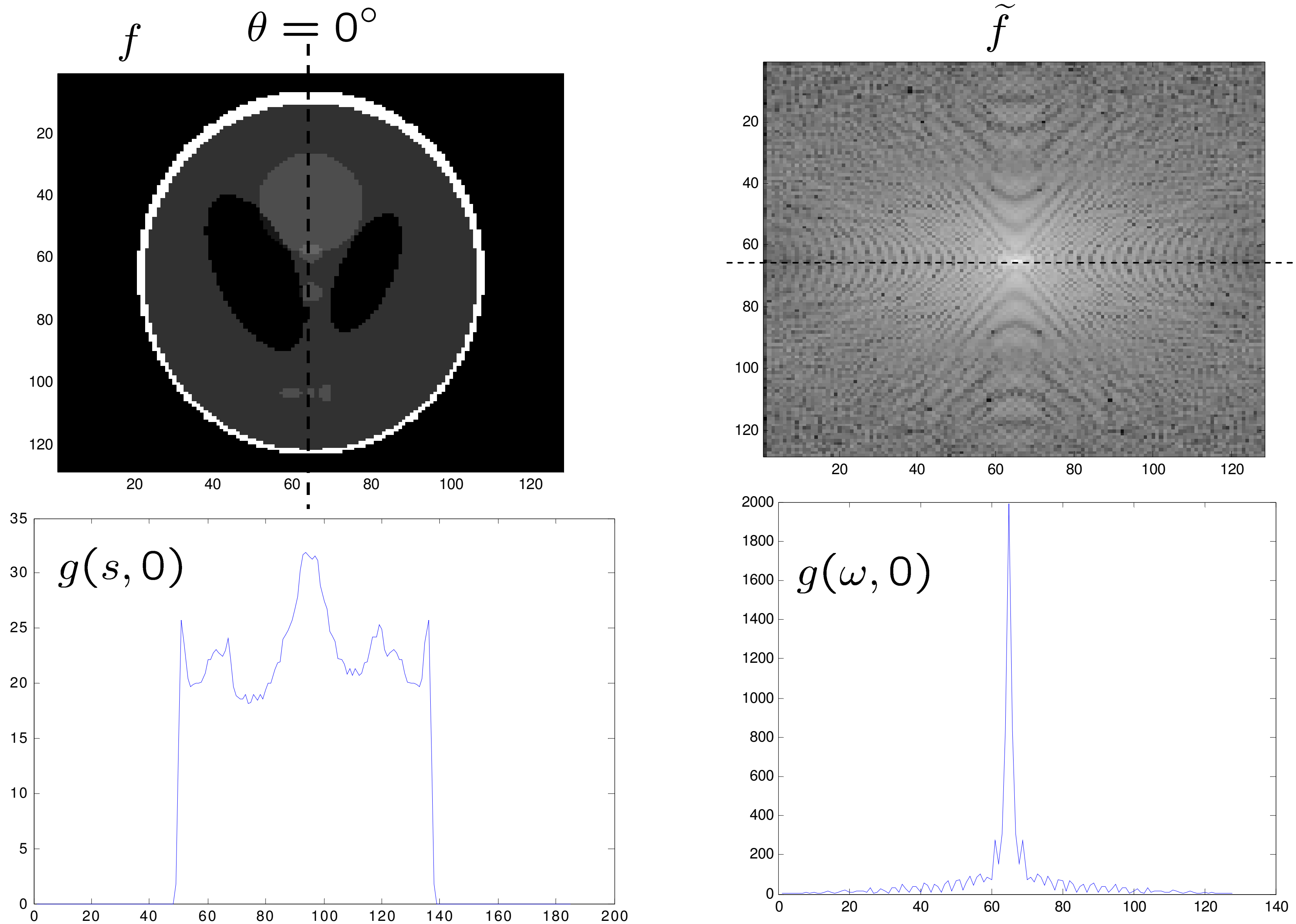
$$= \int f(x)^{-j\omega\,\theta\cdot x}\, dx$$

$$\Rightarrow$$

$$\boxed{\tilde{g}(\omega,\theta) = \tilde{f}(\omega\theta)}$$

$$\tilde{f}(\omega_1,\omega_2) = \int f(x)\, e^{-j(\omega_1,\omega_2)\cdot x}\, dx$$

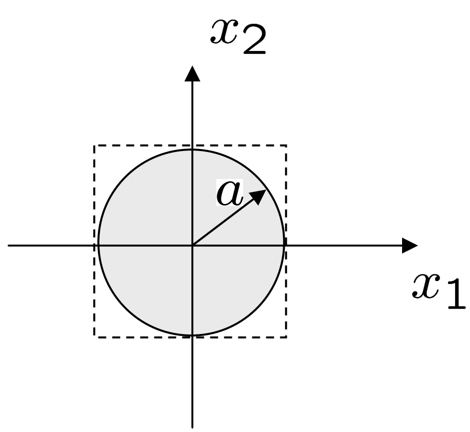
Fourier slice theorem: illustration



Fourier slice theorem: consequences

1- The solution of the inverse problem $Rf = g$, when exists, is unique

2- The knowledge of the the Radon transform of f implies the knowledge of the Fourier transform of f



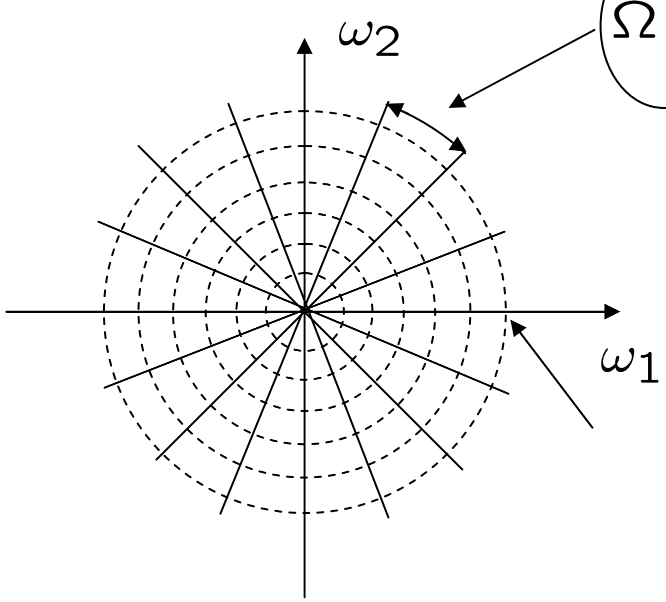
$f(x)$ is spacelimited

$\Rightarrow$

$\tilde{f}(\omega)$ can be computed from samples taken π/a apart

Fourier slice theorem: consequences

p projections



$$\Omega \frac{\pi}{p} \leq \frac{\pi}{a} \quad \Rightarrow \quad \Omega \leq \frac{p}{a}$$

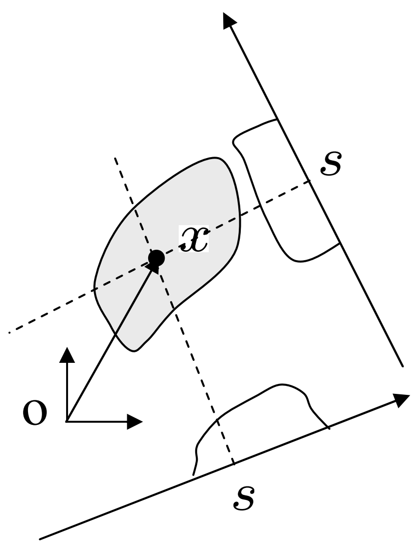
Assumnig that $\tilde{f}(\omega)$ is bandlimited to Ω then it is possible to restore f with a resolution

$$\delta = \frac{2\pi}{2\Omega} = \frac{\pi a}{p}$$

Backprojection operator

$$(R^{\#}g)(x) \equiv \int_0^{2\pi} g(\underbrace{x \cdot \theta}_{s}, \theta)\, d\phi$$

$(R^{\#}Rf)(x)$ is the sum of all line integrals that crosses the point x



Backprojection operator

$R^{\#}$ is the adjoint operator of R when the usual L^2 inner product is considered for both the data functions and the objects

$$\langle Rf, g\rangle_{\mathcal{Y}} = \int_0^{2\pi} \left(\int_{-\infty}^{\infty} (Rf)(s,\theta) g(s,\theta)\, ds \right) d\phi$$

$$= \int_0^{2\pi} \left\{ \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(s\theta + t\theta^{\perp})\, dt \right) g(s,\theta)\, ds \right) d\phi$$

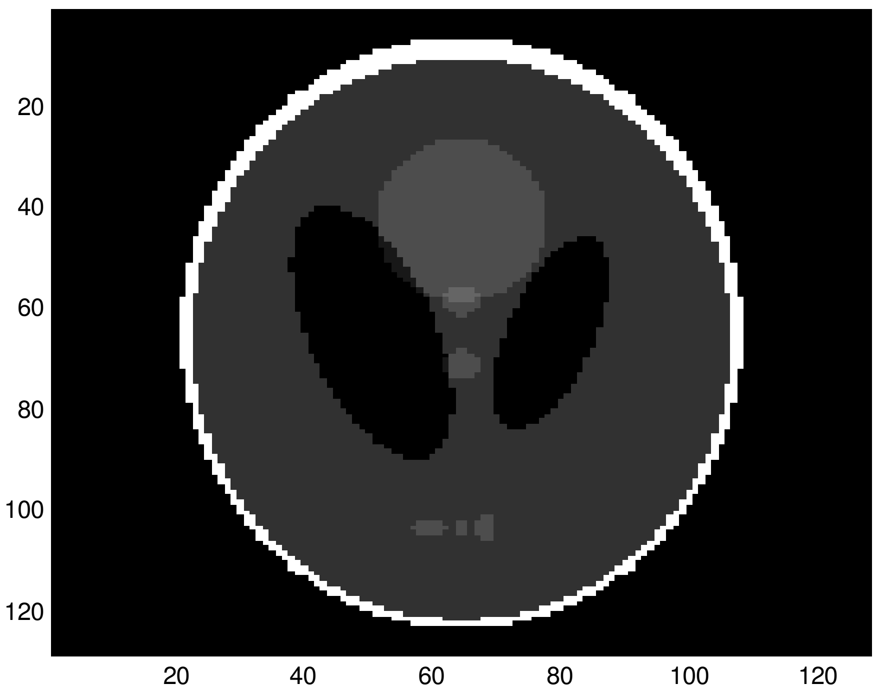
$$= \int_{\mathbb{R}^2} f(x) \left(\int_0^{2\pi} g(x\cdot\theta, \theta)\, d\phi \right) dx$$

$$= \langle f, R^{\#} g\rangle_{\mathcal{X}}$$

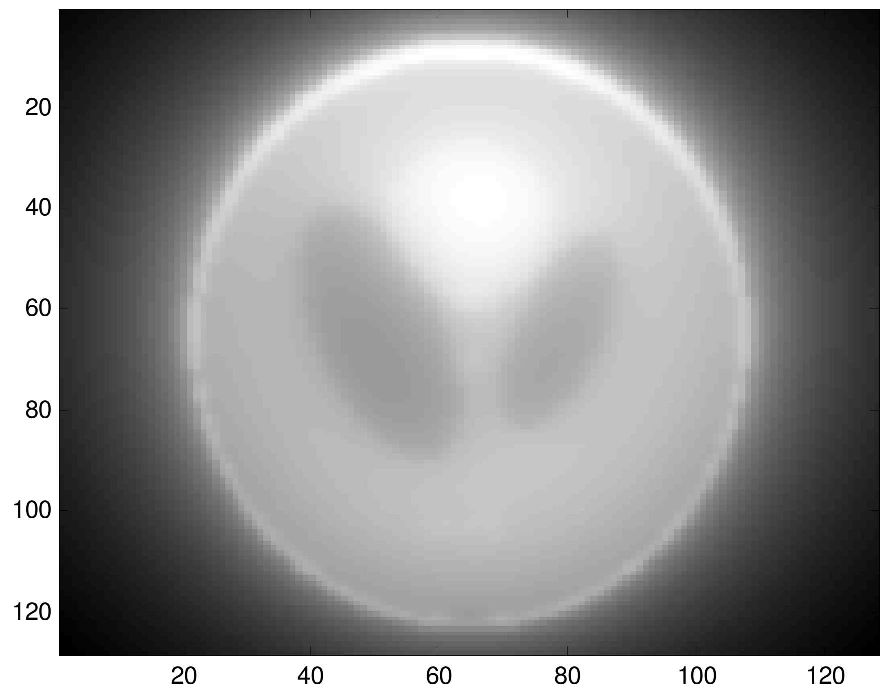
Backprojection operator

$$(R^{\#}Rf)(x) = 2f(x) \star \frac{1}{|x - x'|}$$

f



$R^{\#}Rf$



Filtered backprojection algorithm

$$f(x) = \frac{1}{(2\pi)^2}\int \tilde{f}(\omega)e^{j\omega\cdot x}\,d\omega \qquad \omega = |\omega|\underbrace{(\cos\phi, \sin\phi)}_{\theta}$$

$$f(x) = \frac{1}{(2\pi)^2}\int_0^{2\pi}\left(\int_0^{\infty}|\omega|\underbrace{\tilde{f}(|\omega|\theta)e^{j|\omega|\theta\cdot x}\,d|\omega|}\right)d\phi$$

periodic function of ϕ

$$f(x) = \frac{1}{(2\pi)^2}\int_0^{2\pi}\left(\int_0^{\infty}|\omega|\tilde{f}(-|\omega|\theta)e^{-j|\omega|\theta\cdot x}\,d|\omega|\right)d\phi \qquad \omega' = -|\omega|$$

$$= \frac{1}{(2\pi)^2}\int_0^{2\pi}\left(\int_{-\infty}^{0}|\omega'|\tilde{f}(\omega'\theta)e^{j\omega'\theta\cdot x}\,d\omega'\right)d\phi$$

$$= \frac{1}{2(2\pi)^2}\int_0^{2\pi}\left(\int_{-\infty}^{\infty}|\omega'|\tilde{f}(\omega'\theta)e^{j\omega'\theta\cdot x}\,d\omega'\right)d\phi$$

Filtered backprojection algorithm

$$f(x) = \frac{1}{2(2\pi)^2} \int_0^{2\pi} \left(\int_{-\infty}^{\infty} |\omega'| \tilde{f}(\omega'\theta) e^{j\omega'\theta \cdot x} \, d\omega' \right) d\phi$$

From the Fourier slice theorem $\tilde{f}(\omega\theta) = \tilde{g}(\omega, \theta)$

Defining the filtered backprojections as

$$G(s,\theta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\omega| \tilde{g}(\omega,\theta) e^{j\omega s} \, d\omega$$

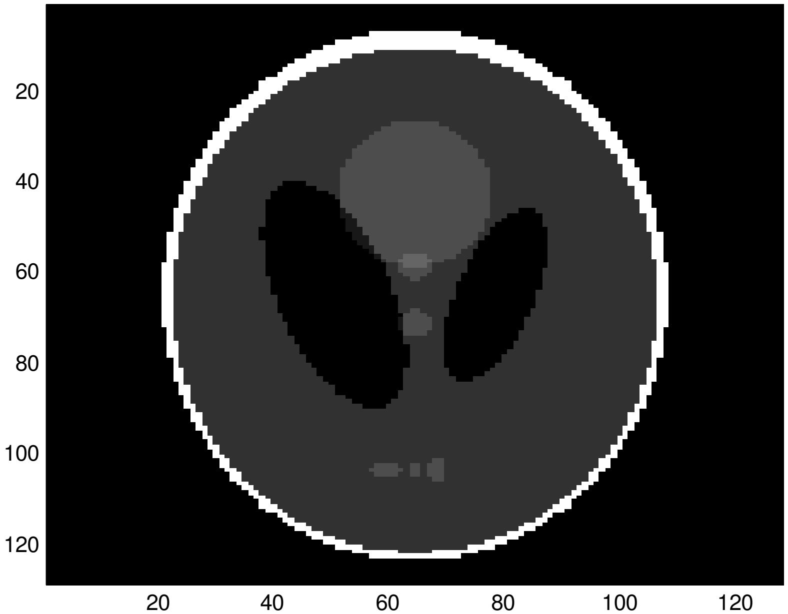
Then

$$f(x) = \frac{1}{4\pi} R^{\#} G$$

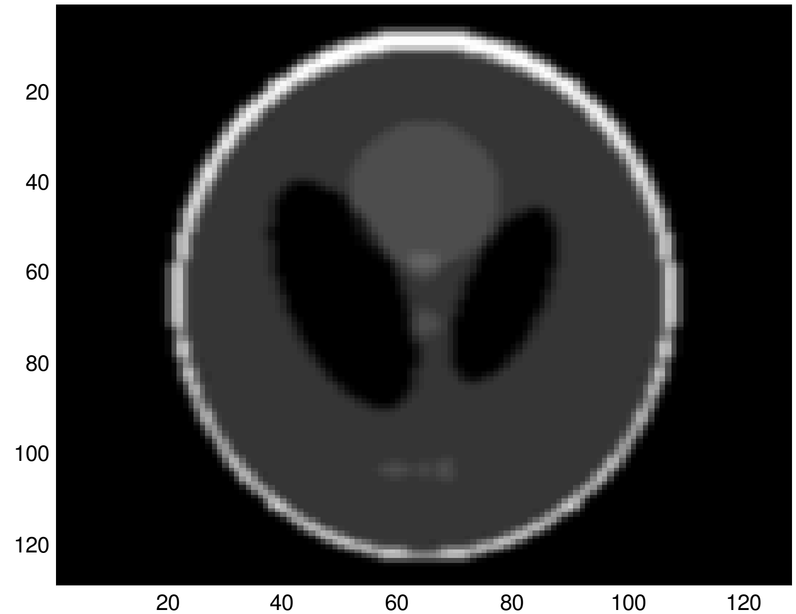
Summary of the filtered backprojection algorithm

- for each value of θ compute the Fourier transform $\hat{g}(\omega, \theta)$ of $g(s, \theta)$
- multiply $\hat{g}(\omega, \theta)$ by the ramp filter $|\omega|$
- compute the inverse Fourier transform of $|\omega|\hat{g}(\omega, \theta)$ to obtain the filtered projections $G(s, \theta)$
- apply the backprojection operator to $G(s, \theta)$.

Filtered backprojection



Hann filter



TV - Reconstruction

Ram-Lak filter