Compressed Sensing

- Compressible/k-sparse signals
- Stable measurements
- Restricted isometric property
- Signal reconstruction algorithms
- Geometric interpretation
- Constrained/Unconstrained optimization
- Fast algorithms

K-sparse signals

$f \in \mathbb{R}^n$ – a signal/image

$\{\psi_i \in \mathbb{R}^n\}_{i=1}^n$ – an orthogonal basis for $\mathbb{R}^n$

$$\Psi \equiv [\psi_1 \ldots, \psi_n] \in \mathbb{R}^{n \times n} \qquad \Psi\Psi^T = I$$

$$f = \sum_{i=1}^{n} \psi_i s_i = \Psi s, \qquad s_i = \langle f, \psi_i \rangle$$

Definition of k-sparse signals:

f is k-sparse if only k coefficients s_i are nonzero

Compressible signals

- There exists a basis where the representation

$$f = \sum_{i=1}^{n} \psi_i s_i$$

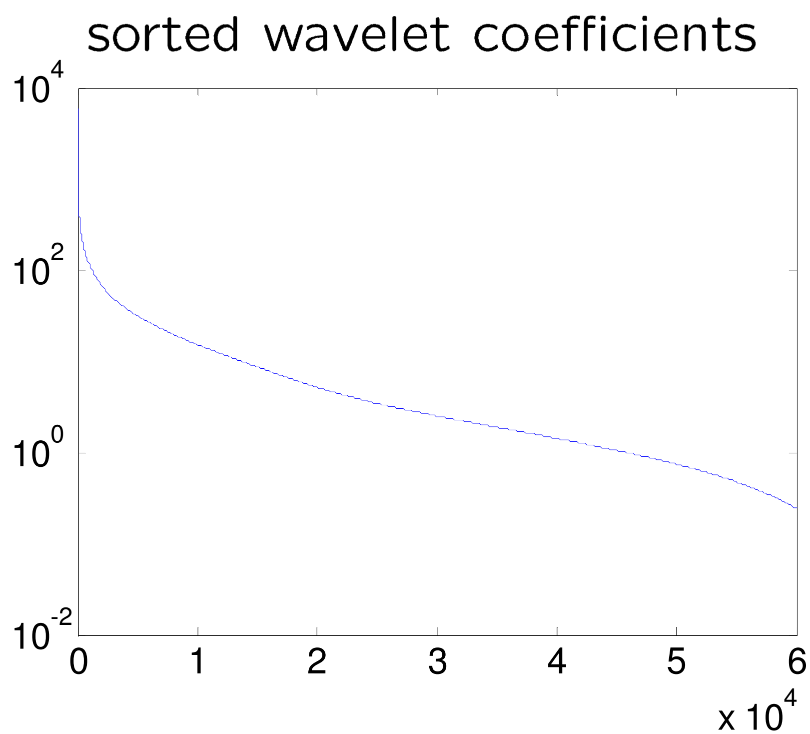
has just a few large coefficients and many small coefficients.

- Compressible signals are well approximated by k-sparse representations

- Many natural and man-made signals are compressible

Example: wavelet representation

256×256 image



Example: wavelet representation (cont.)

original image

f_k approximation

$k = 0.1n$

$$\|f - f_k\|_2 \leq 0.08 \, \|f\|_2$$

Classical transform coding of compressive signals

1. Acquire the full n-sample signal
2. Compute the set of n coefficients $s_i = \langle f, \psi_i \rangle$
3. Retain the k << n largest coefficients and encode their locations and values

Shortcommings:

1. Acquire n samples and only k << n coefficients are retained
2. The encoder computes n coefficients
3. The locations, besides the values of the largest k coefficients, have to be encoded

Goal of compressive sampling

$$g = \Phi f \in \mathbb{R}^m \qquad m < n \qquad g_i = \langle \phi_i, f \rangle$$

↓

Measurement matrix

$$\Theta \equiv \Phi\Psi \in \mathbb{R}^{m\times n} \qquad g = \Theta s$$

Goal of CS

Design a measurement matrix Φ and a reconstruction algorithm for k-sparse and compressible signals such that k is of the order of m

Stable measurement matrix

The measurement processes must not erase the information present in the k non-zero entries of signal f

Restricted isometric property (RIP)

For any k-sparse vector f there holds

$$1 - \varepsilon \leq \frac{\|\Theta f\|_2}{\|f\|_2} \leq 1 + \epsilon$$

for some $\varepsilon > 0$

Any $m \times k$ submatrix of Θ is well-conditioned

Another way to look at stable measurements

The measurement matrix ϕ must be incoherent with the sparsifying basis Ψ in the sense that the vectors ϕ_i cannot sparsely represent the vectors ψ_j and vice versa.

Stable (in the RIP sense) measurement matrices

$\phi_{ij} \equiv$ random Gaussian

$\phi_{ij} \equiv$ random binary

$\phi_{ij} \equiv$ randomly selected Fourier samples (extra log factors apply)

Signal reconstruction algorithms

Minimum ℓ_0-norm reconstruction

$$\hat{s} = \arg\min_s \|s\|_0 \qquad \text{subject to} \quad \Theta s = g$$

NP- complete problem !!!!

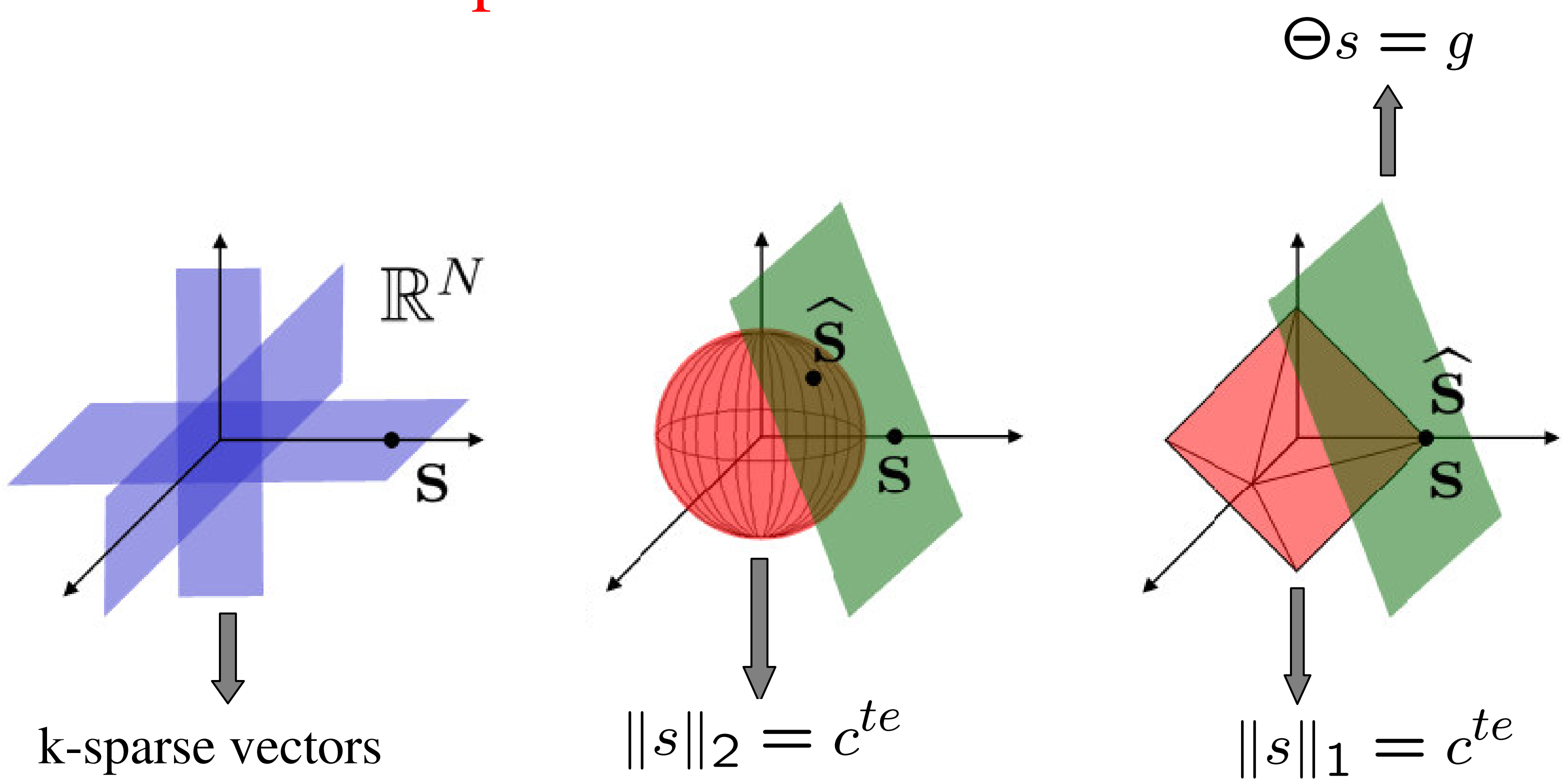
(Convex approximation) Replace ℓ_0 with ℓ_1

$$\hat{s} = \arg\min_s \|s\|_1 \qquad \text{subject to} \quad \Theta s = g$$

Basis pursuit problem ($O(n^3)$ complexity)

Surprise: if the measurement matrix is Gaussian i.i.d, then a k-sparse vector f is exactly recovered by solving the ℓ_1 optimization problem with $m = ck\log(n/k)$ observations

Geometrical interpretation



Unconstrained optimization

Minimum ℓ_1 -norm reconstruction

$$\widehat{s} = \arg\min_s \|s\|_1 \qquad \text{subject to} \quad \Theta s = g$$

Equivalent

$$\widehat{s} = \arg\min_s \|\Theta s - g\|_2^2 + \lambda \|s\|_1$$

Basis denoising algorithm

Algorithms

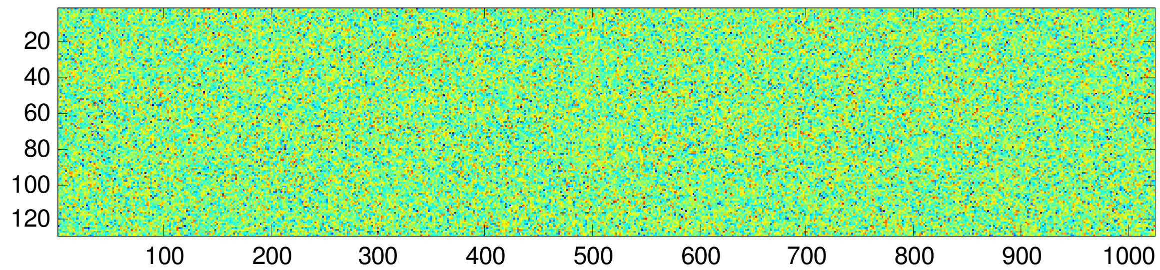
- l1_ls [Kim *et al*, 2007]
- GPRS [Figueiredo et al, 2007]
- TwIST [Bioucas-Dias & Figueiredo , 2007]

Example: Sparse reconstruction

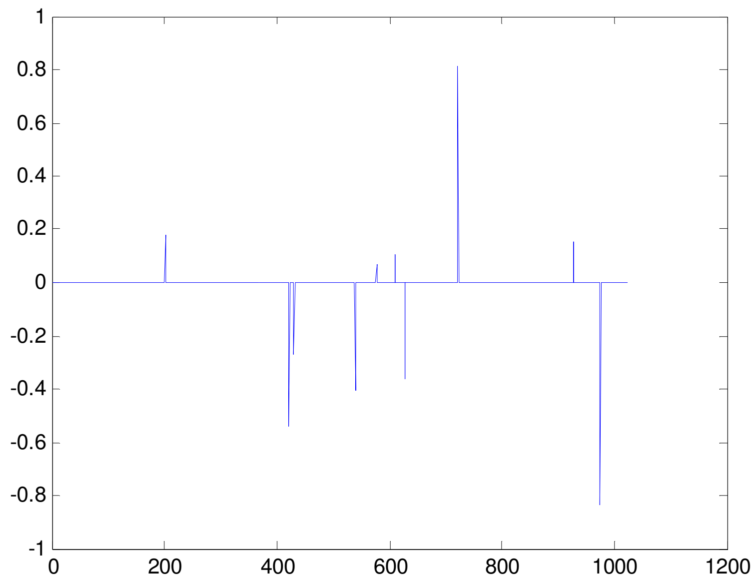
$$L_\alpha(f) = \|g - \Phi f\|^2 + \alpha\|f\|_1$$

Φ is i.i.d. Gaussian

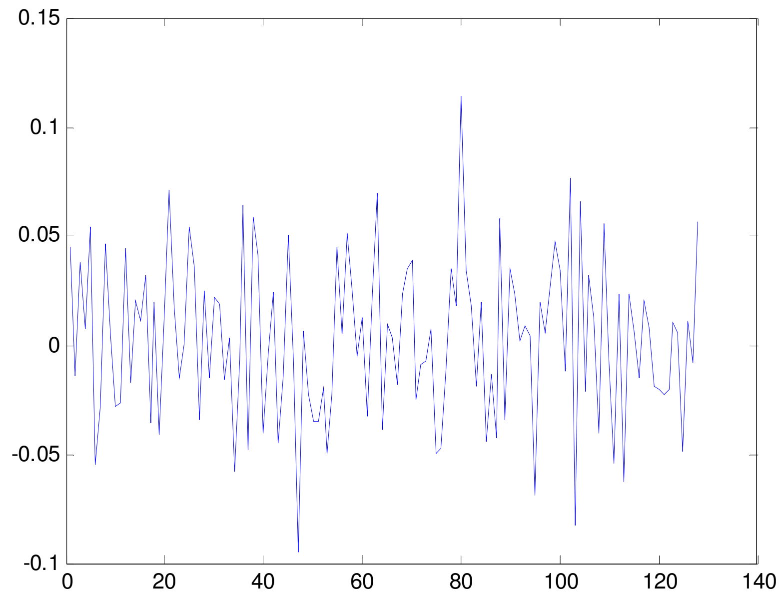
Φ



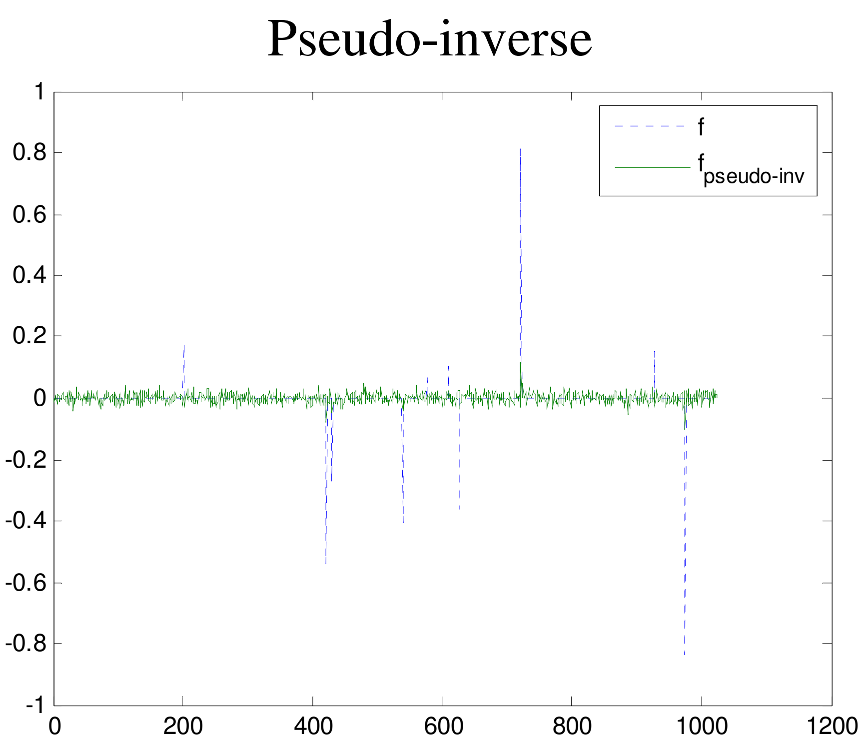
Original data - f

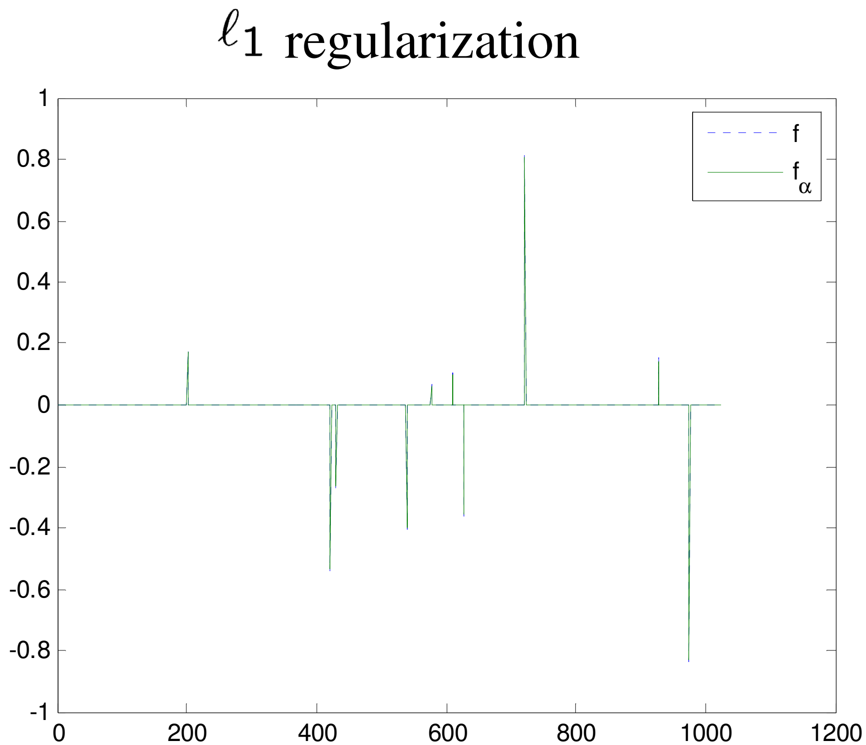


Observed data - g



Example: Sparse reconstruction

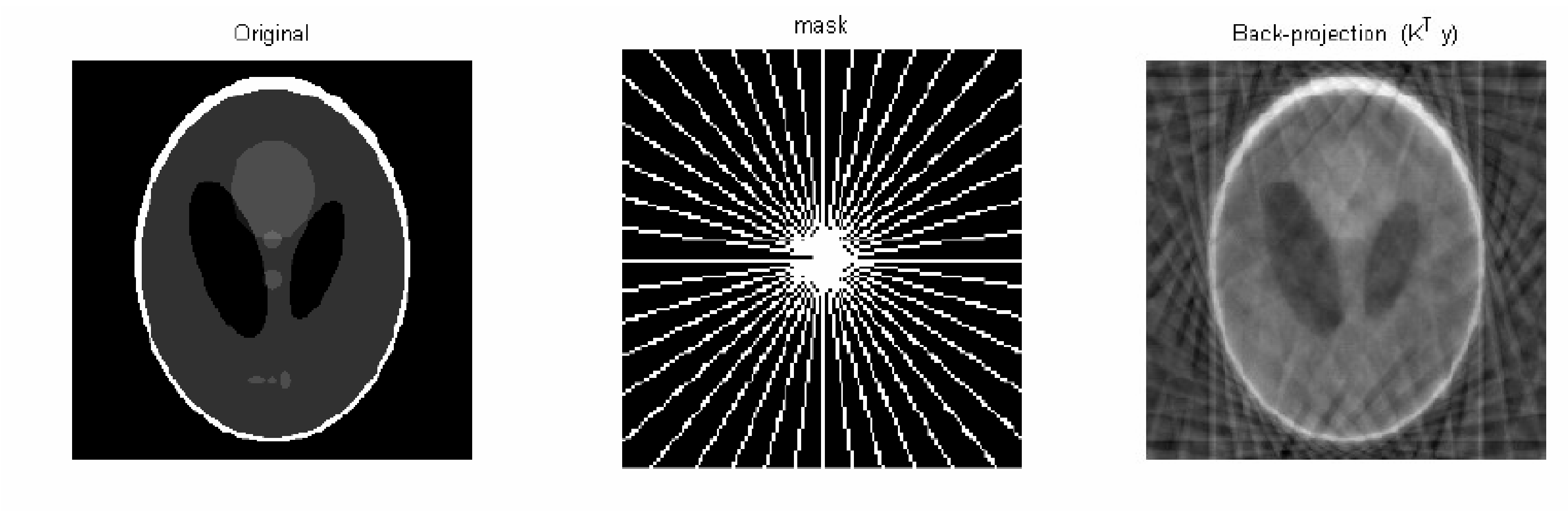




Example: Total variation reconstruction

$$L_\alpha(f) = \|g - \Phi f\|^2 + \alpha \mathsf{TV}(f) \qquad \mathsf{TV}(f) = \sum_i \sqrt{(\Delta_i^h f + \Delta_i^v f)}$$

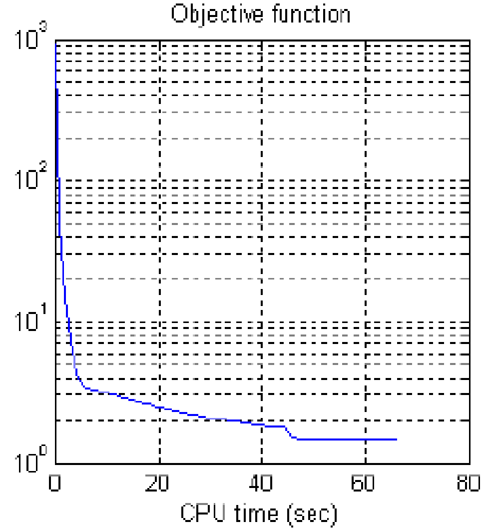
Φ randomly selected Fourier samples (9%)

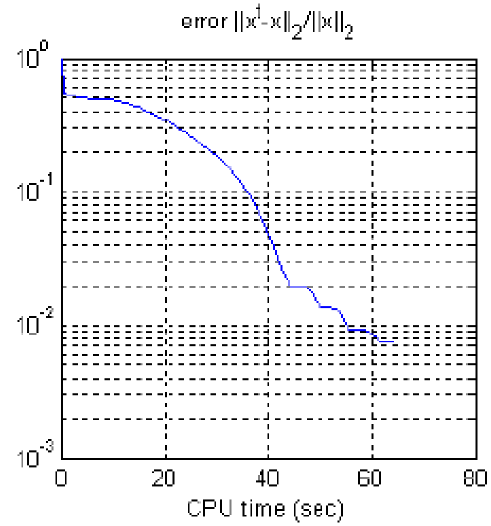


Original

Algorithm	Time (sec)
TwIST	66
IST	937
tveq logbarrier (l1_magic)	4827

Estimate





References

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