BAND STRUCTURE OF MEDIA WITH HIGHLY LOCALIZED PERMITTIVITY DISTRIBUTIONS

Mário G. Silveirinha [1], **Carlos A. Fernandes**[2]

[1] Departamento de Engenharia Electrotécnica e Computadores da Universidade de Coimbra, 3030, Coimbra, Portugal, mario.silveirinha@co.it.pt
[2] Instituto Superior Técnico, Av. Rovisco Pais 1049-001 Lisboa, Portugal, carlos.fernandes@lx.it.pt

Abstract – In this paper we propose a problem that concerns the calculation of the band structure of periodic media with delta-function permittivity distributions. The objective is to obtain an analytical solution for the dispersion characteristic, or alternatively an efficient numerical solution. The proposed problem is in a certain sense canonical, and is of interest to understand how the propagation of electromagnetic waves in periodic structures depends on the medium topology.

Keywords – Photonic Crystals, 3D Wire Medium, Eigenvalue Problems.

I. INTRODUCTION

In recent years, the engineering of composite periodic media has experienced an extraordinary impulse after the discovery of electromagnetic band gap structures [1]-[3], and more recently of left-handed media [4]. The emergence of new "engineered" materials is expected to have a great impact in the performance of antennas and other devices for radio and millimeter wave applications. Artificial materials consist of a host medium with metallic or dielectric implants.

The problem proposed in this paper is connected to the calculation of the dispersion characteristic of periodic dielectric materials.

The propagation of electromagnetic waves is described by the Maxwell-Equations [5]. In periodic structures, a generic solution of the Maxwell-Equations can be decomposed into Floquet solutions, which are characterized by a wave vector **k**.

For a given wave vector, the Floquet solutions occur only for a numerable set of "resonant frequencies", β_n, n=1,2.... The band structure of the periodic medium is formed by the multi-valued application $\mathbf{k} \rightarrow \beta_n(\mathbf{k})$ [2].

The calculation of the band structure of periodic media is an intricate problem. Mathematically the problem can be reduced to the calculation of the spectrum of a Hermitian operator. Several methods have been proposed for the effect [2]-[3], [6]-[7]. The most popular method is perhaps the plane wave method [7], which, simply put, consists in expanding the pertinent physical quantities in a Fourier series, reducing in this way the eigenvalue problem to the calculation of the spectrum of an Hermitian matrix.

The numerical calculation of the band structure of periodic materials, although computationally demanding, is well established. In this paper, we propose the study of a new class of media of highly localized delta-function permittivity distributions.

Although this class of media is not physically realizable, its band structure may provide valuable information concerning the effect of the medium topology on the propagation of electromagnetic waves. The interest in introducing this problem is related to the fact that problems involving delta functions are more likely to have analytical solutions than problems with stepwise continuous permittivity distributions.

II. EXAMPLES

In this section we present examples that illustrate the objective of the paper with two particular cases.

A. 1-D problem

First we consider a one-dimensional problem. In this case the objective is to determine the Floquet modes of the following equation:

$$\frac{d^2\psi}{dx^2} + \beta^2 \varepsilon(x)\psi = 0 \qquad (1)$$

where $\varepsilon(x)$ is the periodic (relative permittivity), $\psi = \psi(x)$ is the wave function, and β is the "wave number". The permittivity satisfies:

$$\varepsilon(x+a) = \varepsilon(x), \quad \text{generic } x \qquad (2)$$

where a is the period. We assume that the permittivity is given by:

$$\varepsilon(x) = 1 + (\varepsilon_{av} - 1) a \delta_p^{(1)}(x) \qquad (3)$$

$$\delta_p^{(1)}(x) = \sum_n \delta(x - na) \qquad (4)$$

where δ is Dirac's distribution, ε_{av} is the average permittivity, and *n* is a generic integer.

We look for Floquet solutions of equation (1), that is for solutions such that:

$$\psi(x)\exp(j\,kx) \text{ is periodic} \tag{4}$$

where $j=\sqrt{-1}$ and *k* is the "wave vector" (the wave vector is a given real number that can always be assumed in $[-\pi/a,\pi/a]$: the Brillouin zone [2], [3], [8]). The objective is to determine the wave numbers β, such that (1) subject to (3) has a non-trivial solution.

The solution of the enunciated problem can be easily obtained. We can for example approximate the periodic delta distribution in (4) by sequence of stepwise constant functions, and reduce the problem to Hill's equation [8]. In this way, we conclude that for a given *k*, equation (1) has non-trivial solutions for β such that:

$$F(\beta,k,\varepsilon_{av}-1)=0 \tag{5}$$

where,

$$F(\beta,k,u)=-\cos(ka)+\cos(\beta a)-\frac{1}{2}u\,\beta a\sin(\beta a) \tag{6}$$

The particular case in which we let ε_{av} approach infinity is particularly relevant, since it corresponds to the analog of a periodic medium with metallic inclusions. In this case, the solutions of (5) are:

$$\beta^2(k)=\left(\frac{n\pi}{a}\right)^2, \qquad n=1,2,... \tag{7}$$

Notice that the right-hand side of the above equation is independent of *k*. The medium is thus "dispersionless". The calculated resonant frequencies are unsurprisingly coincident with those of a metallic cavity.

B. 3-D scalar problem

We consider now an example of the three-dimensional scalar problem. The objective is to calculate the Floquet solutions of the following equation:

$$\nabla^2\psi+\beta^2\varepsilon(\mathbf{r})\psi=0 \tag{8}$$

where ∇^2 is the Laplacian and $\mathbf{r}=(x,y,z)$. The permittivity $\varepsilon(\mathbf{r})$ is periodic in both *x*, *y* and *z*. We take it equal to:

$$\varepsilon(\mathbf{r})=1+\frac{(\varepsilon_{av}-1)a}{3}\left[\delta_p^{(1)}(x)+\delta_p^{(1)}(y)+\delta_p^{(1)}(z)\right] \tag{9}$$

This topology corresponds to the situation in which high permittivity regions surround low-permittivity "islands" (the high permittivity region is connected). We look for solutions of equation (8) such that:

$$\psi(\mathbf{r})\exp(j\,\mathbf{k.r}) \text{ is periodic} \tag{10}$$

where $\mathbf{k}=(k_x,k_y,k_z)$ is a given wave vector (k_x, k_y, and k_z are real numbers that can always be assumed in $[-\pi/a,\pi/a]$).

Since equation (8) is separable, we easily conclude by putting $\psi=\psi_1(x)\psi_2(y)\psi_3(z)$ that the sought dispersion characteristic satisfies:

$$\beta^2=\beta_1^2+\beta_2^2+\beta_3^2 \tag{11.a}$$

$$F\left(\beta_1,k_x,\frac{1}{3}\frac{\beta^2}{\beta_1^2}(\varepsilon_{av}-1)\right)=0 \tag{11.b}$$

$$F\left(\beta_2,k_y,\frac{1}{3}\frac{\beta^2}{\beta_2^2}(\varepsilon_{av}-1)\right)=0 \tag{11.c}$$

$$F\left(\beta_3,k_z,\frac{1}{3}\frac{\beta^2}{\beta_3^2}(\varepsilon_{av}-1)\right)=0 \tag{11.d}$$

If we let ε_{av} approach infinity, it clear that the solution of the above system converges to:

$$\beta^2(\mathbf{k})=\left(\frac{n_1\pi}{a}\right)^2+\left(\frac{n_2\pi}{a}\right)^2+\left(\frac{n_3\pi}{a}\right)^2 \tag{12}$$

where n_1, n_2 and n_3 are ≥ 0 and $(n_1,n_2,n_3)\neq(0,0,0)$. As in the previous section, if ε_{av} goes to infinity the medium becomes dispersionless, and the spectrum consists of a set of flat bands. Physically this is easily understood, since the low permittivity regions become metallic cavities.

III. THE PROBLEM

The problem proposed here is to obtain the band structure of the scalar equation (8), or preferably of Maxwell-Equations (in the frequency domain):

$$\nabla^2\mathbf{E}+\beta^2\varepsilon(\mathbf{r})\mathbf{E}=0 \tag{13.a}$$

$$\nabla.\mathbf{E}=0 \tag{13.b}$$

where $\mathbf{E}=(E_x,E_y,E_z)$ is the electric field, and $\nabla.\mathbf{E}$ is the divergence. Analytical solutions are sought. Alternatively, an efficient numerical solution is also of interest. The important case is that in which ε_{av} approaches infinity.

The relevant permittivity distributions are described in next sections.

A. "Spot Media"

In the "spot medium" the low-permittivity region is connected. This structure is complementary from that analyzed in section II.B. The permittivity is given by:

$$\varepsilon(\mathbf{r}) = 1 + (\varepsilon_{av} - 1)a^3 \delta_p^{(3)}(\mathbf{r}) \tag{14}$$

$$\delta_p^{(3)}(\mathbf{r}) = \sum_{n,m,l} \delta(x - na)\delta(y - ma)\delta(z - la) \tag{15}$$

where *n, m* and *l* are generic integers.

B. "Wire Media"

The "wire medium" is characterized by "long-thin" high - permittivity regions (wires or veins) [9]. The corresponding permittivity is given by:

$$\varepsilon(x, y, z) = 1 + (\varepsilon_{av} - 1)a^2 \delta_p^{(2)}(x, y) \tag{16}$$

$$\delta_p^{(2)}(x, y) = \sum_{n,m,l} \delta(x - na)\delta(y - ma) \tag{17}$$

C. "2D-Wire Media"

In the "2D-wire" medium the wires are oriented in two possible orthogonal directions:

$$\varepsilon(x, y, z) = 1 + (\varepsilon_{av} - 1)a^2 \frac{1}{2}\left[\delta_p^{(2)}(x, y) + \delta_p^{(2)}(y, z)\right] \tag{18}$$

D. "3D-Wire Media"

In the "3D-wire" medium [10] the wires are oriented in three possible orthogonal directions:

$$\varepsilon(x, y, z) = 1 + (\varepsilon_{av} - 1)a^2 \frac{1}{3}\left[\delta_p^{(2)}(x, y) + \delta_p^{(2)}(y, z) + \delta_p^{(2)}(x, z)\right] \tag{19}$$

REFERENCES

[1] E. Yablonovitch, "Inhibited Spontaneous Emission in Solid-State Physics and Electronics", Phys. Rev. Letts 58, pp.2059, May 1987

[2] J. Joannopoulos, R. Meade, J. Winn, *Photonic Crystals*, Princeton University Press, 1995

[3] K. Sakoda, *Optical Properties of Photonic Crystals*, Springer Series in Optical Sciences 80, 2001.

[4] D.R. Smith, W.J. Padilla, D.C. Vier, S.C. Nemat-Nasser, S. Schultz, "Composite Medium with Simultaneously Negative Permeability and Permittivity", Phys. Rev. Letts. 84, pp.4184, May 2000

[5] R.E. Collin, *Field Theory of Guided Waves*, 2nd Ed., IEEE Press, 1991

[6] K.M. Ho, C.T. Chan, C.M. Soukoulis, "Existence of a Photonic Gap in Periodic Dielectric Structures", Phys. Rev. Letts. 65, pp.3152, Dec. 1990

[7] M. Silveirinha, C. A. Fernandes, "Efficient Calculation of the Band Structure of Artificial Materials with Cylindrical Metallic Inclusions", IEEE Trans. on MTT May 2003

[8] L. Brillouin, *Wave Propagation in Periodic Structures*, 2nd Ed., Dover Publications, 1953

[9] C. A. Moses, N. Engheta, "Electromagnetic wave propagation in the wire medium: a complex medium with long thin inclusions", Wave Motion vol. 34, pp. 301-317, 2001

[10] D.F. Sievenpiper, M.E. Sickmiller, E. Yablanovitch, "3D Wire Mesh Photonic Crystals", Phys. Rev. Letts., vol.-76 pp.2480, 1996