Finding a stability region for a congestion control algorithm

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Abstract— We present a problem coming from the area of congestion control in packet-switching networks. The theme is the assessment of the stability regions of a time-invariant discrete-time system. The problem is to find the conditions upon which a given polynomial, whose coefficients are subject to certain restrictions, has all roots inside the unit disc.

Keywords— Linear control systems, stability regions, roots of polynomials.

I. Introduction

We focus on congestion control mechanisms for packet-switching networks, looking at schemes where the sources of traffic receive congestion feedback from the network. One of the key components for the efficient control of resources in a large scale network is the provision of an active mechanism at the core routers (called *router algorithm*), which is able to monitoring the current state of congestion and react accordingly by sending feedback messages to the sources. There are many algorithms that have been proposed under different design objectives and network assumptions, resulting in different control strategies. In spite of that, a common concern in the design of these algorithms is the assessment of their stability regions, either through simulation studies or through a suitable analytic model that captures the essential characteristics of the overall system. In one such proposal, the routers perform control actions without keeping per-flow information (a particular strategy known as *stateless* congestion control) and update an internal value p_k at discrete epochs k. The value p_k represents the estimated fair share of bandwidth. The values of p_k are communicated back to the sources. At each router, p_k is updated according to the following relation:

$$p_k = \gamma\, p_{k-1} + \alpha_k\, \left(U_k - R_k\right)\,, \tag{1}$$

where γ and α_k are control parameters, U_k is the available capacity of the router and R_k is the measured total input traffic of the router. The sources are responsive and react to the (delayed) messages p_k sent by their bottleneck routers. This way, the measured traffic component R_k can be modeled as a weighted sum of past values on p_k, up to a maximum (discrete) delay n on the network, as follows:

$$R_k \,=\, a_1(k)\, p_{k-1} + a_2(k)\, p_{k-2} + \cdots + a_n(k)\, p_{k-n}\,. \tag{2}$$

The weight factors $a_i(k)$ depend on the distribution of sources in the network and are, in general, time-variant parameters. The overall interaction among each bottleneck router and sources can be modeled as a discrete-time control system with delayed feedback.

In general, the resulting discrete-time system is subject to variations on the parameters $a_i(k)$ (due to the dynamics of sources entering and leaving the network) and, therefore, the control parameter α_k must be evaluated at each epoch k in order to drive the system into stable regions. The router has no means to determine the individual components $a_i(k)$ but, instead, is able to estimate an upper bound on $\sum_i a_i(k)$, denoted by $M(k)$, which represents the number of locally restricted sources. Even though the distribution of the parameters $a_i(k)$ is unknown to the router, we can apply general results on the stability of time-variant systems to derive stability regions for the algorithm, using the estimated values of $M(k)$ and imposing conditions on the parameter γ and control gain α_k.

II. Time-invariant case

We now turn our attention to the special case where the coefficients $a_i(k)$ are *invariant*, i.e. $a_i(k) = a_i$. The estimated upper bound on the number of locally restricted sources is now an invariant parameter, given by $M = \sum_i a_i$. In some network scenarios, it is not unrealistic to assume that the variation on the population of sources competing for bandwidth is very slow as compared to the evaluation intervals of the routers. In addition, we are interested in the particular case where $\gamma = 1$. This turns the previous model into a time-invariant discrete-time system, whose stability can now be assessed by looking at the roots of the resulting characteristic polynomial:

$$P(Z) \,=\, Z^n + (\alpha\, a_1 - 1) Z^{n-1} + \alpha\, a_2 Z^{n-2} + \cdots + \alpha\, a_n\,. \tag{3}$$

The system is stable if and only if $P(Z)$ has all zeros inside the unit disc. Under the time-invariant case, we may look further at particular distributions on the parameters a_i which have strong connections to practical network scenarios. For example, we may consider a worst-case scenario for design purposes, where it is assumed that all control delays in the network are concentrated in the parameter a_n. In other words, this means that all sources in the network react with the maximum delay of the network. This particular distribution leads to the characteristic polynomial

$$P(Z) \, = \, Z^n - Z^{n-1} + \mu \qquad\qquad n \geq 2 \, , \tag{4}$$

where

$$\mu \, = \, \alpha \, a_n \, = \, \alpha \, M \, .$$

It can be proved that the polynomial (4) has all its zeros inside the unit disc if

$$\mu \, < \, 2 \, \sin \frac{\pi}{2(2\,n-1)} \, , \tag{5}$$

which can be re-written as:

$$\alpha \, < \, \frac{2}{M} \, \sin \frac{\pi}{2(2\,n-1)} \, . \tag{6}$$

Now, let us assume that the system is set according to (6) (i.e. considering a worst-case scenario) but the network is subject to intermediate configurations, bound to the maximum delay n, such that

$$P(Z) \, = \, Z^n + (\alpha \, a_1 - 1) Z^{n-1} + \alpha \, a_2 Z^{n-2} + \cdots + \alpha \, a_n \, , \tag{7}$$

where a_i are arbitrary nonnegative coefficients and $\sum_i a_i = M$.

The question is: is condition (6) sufficient for polynomial (7) having all its zeros inside the unit disc ? We did not find a formal proof for $n > 2$, although we suspect that the answer is yes. Through numerical analysis we could not find any counter-example.