EFFICIENT AND ACCURATE NUMERICAL SOLUTION OF STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

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Abstract – Accurate numerical computation of a set of stochastic partial differential equations presents considerable difficulties. A technique to deal with this problem in a time efficient way is sought.

Keywords – stochastic differential equation, numerical solution, partial differential equation, nonlinear differential equation, numerical solution accuracy, time efficiency.

I. INTRODUCTION

Semiconductor optical amplifiers and semiconductor distributed feedback lasers are key components for optical networks. The dynamics of those devices, including noise features, is usually described by a set of stochastic partial differential equations (SPDEs), see for instances [1]. The accurate noise characterisation of these devices is of particular interest from a telecommunications point of view or, by other words, the accurate statistical characterisation of the stochastic process at the device output. The accuracy of a numerical scheme for integrating a set of SPDEs is judged on the basis of its ability to provide samples of the stochastic process from which accurate estimates of some statistical parameters can be computed.

In the general case, each equation of the set of SPDEs is nonlinear. Therefore, the rigorous characterisation of noise at the device output can only be obtained by accurately solving, in a numerical way, the set of SPDEs. Often, we are mainly interested in the estimation of moments, probabilities or other functional as the noise power spectral density from samples of the solution of the set of SPDEs. Furthermore, it is also particularly important to obtain accurate estimates in a short time interval because these estimates are often used in a more general process of system optimisation. So, a time efficient and accurate technique of numerically solving a set of stochastic partial differential equations is very desirable and, to the author's knowledge, the development of such technique still remains.

Techniques based on first order approximation in the step size of each independent variable [2] should be avoided because of the high time required to assess each sample. Furthermore, the accuracy, as described above, of the techniques presented in reference [2] is questionable. Techniques like those presented in [2] seem to be particularly devoted to deal with another problem of numerical solution of a set of SPDEs, namely its robustness.

II. FORMULATION OF THE PROBLEM

The problem may be formulated as the derivation of a procedure for accurate numerical integration of the following set of stochastic partial differential equations (propagation equations of the fields inside the device)

$$\frac{\partial A_i(z,t)}{\partial z} + k_i \frac{\partial A_i(z,t)}{\partial t} = f_i(A_1, \dots, A_n, z, t) + \eta_i(z,t) \qquad (1)$$
$$0 \le z \le L, \quad t \ge 0, \quad 1 \le i \le n$$

where the index i refers to the different complex fields with a total count of n, the independent variables corresponding to time and space coordinates are t and z, respectively, L is the maximum space coordinate of interest (device length), $A_i(z, t)$ is the ith complex field, k_i is a constant, $f_i(A_1, \dots, A_n, z, t)$ is a complex nonlinear function of the complex fields representing the nonlinear field evolution, and $\eta_i(z, t)$ is a complex Gaussian-distributed stochastic field. The two Gaussian components of the complex field $\eta_i(z, t)$ are statistically independent, and the stochastic fields $\eta_i(z, t)$ are generally uncorrelated in t and z, so that they satisfy the following property:

$$\left\langle \eta_i(z,t)\eta_i^*(z',t') \right\rangle = \xi_i \delta(t-t')\delta(z-z') \qquad 1 \le i \le n$$

where $\langle x \rangle$ means expected value of x, ξ_i is constant, and $\delta(x)$ is the delta Dirac function.

The procedure should generate representative values of $A_i(z,t)$ at discrete times t_j for the specific $z=L$ by direct solution of the SPDEs. These values of $A_i(z, t)$ are then used to estimate accurately the statistical parameters of interest, as those above mentioned. Preferably, the procedure should produce results that are statistically correct to a given order in the time and space step. Higher order approximations, if they exist, seem desirable because of shorter computation time.

III. A PARTICULAR CASE AND A SUGGESTION

A very interesting and useful technique to solve a set of stochastic differential equations (SDEs) was presented in [3]. The set of SDEs can be seen as a particular case of (1) where the space dependence does not exist. The technique is an extension of the Runge-Kutta method for numerical solution of deterministic differential equations. The main idea of the technique is to evaluate the nonlinear function at stochastically selected points, so that all moments of the extrapolated estimate after a time step are correct to some order in the step size. This extension of the Runge-Kutta method for SDEs, with some modifications, was used to obtain samples of the electric field at a single-mode bulk laser output, as described in [4], [5]. It was proved to be simultaneously very efficient and accurate in the power spectral density estimation of intensity and frequency noises of the field at the laser output and after transmission along a single-mode fibre [4], [5]. Its accuracy was also confirmed for the probability density function of the intensity noise at the laser output.

A generalisation of the Runge-Kutta method to SPDEs or the development of a completely new technique with similar features, regarding accuracy, time efficiency and complexity, would be very desirable.

IV. REFERENCES

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