Depolarization Dyadics

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Abstract

Important ingredients in both exact and approximative treatments that aim to establish the electromagnetic response of an individual scatterer are so–called *depolarization dyadics*. Here, the focus shall be on both the derivation of closed–form expressions for as well as numerical evaluations of the depolarization dyadics. It will be shown how — from the initial approaches to the subject for *isotropic* mediums — important results have been derived for the general *anisotropic* and general *bianisotropic* regime in recent years.

1. Introduction

In order to motivate our approach to the topic of depolarization dyadics, we shall consider an applicational area of electromagnetics in which they play a significant role. That topic is *homogenization*, where two or more component mediums (often envisioned in particulate form) are mixed together to form a composite material. The aim of homogenization theories is then to extract the constitutive parameters of the composite material from a knowledge of the constitutive and geometrical properties of the constituent mediums. The research literature on homogenization is vast. The interested reader will find many important historical works in an anthology [1] and a comprehensive review of new developments in the homogenization of linear bianisotropic materials in a recent book chapter [2].

The first step towards the derivation/estimation of the composite's constitutive properties requires the solution of an electromagnetic scattering problem in which the electromagnetic response of an individual scatterer must be determined. This task can be achieved by adopting *direct* or *indirect scattering approaches*. For a detailed discussion of merits, difficulties and limitations of these approaches, the reader is referred to an up–to–date conceptual review by Lakhtakia [3]. While the direct scattering approach usually consists of numerical implementations, some analytical progress can often be made in the indirect scattering approaches, especially when certain approximations are introduced where appropriate. In particular, in the so–called *Rayleigh estimate*, the complete information about the electromagnetic response is contained in the depolarization dyadics.

In the present review the topic of depolarization dyadics is developed through a complete and exact representation of the electromagnetic field that is applicable even when the field point is contained within the source region. Alternatively, it is shown how the depolarization dyadics of a general, linear bianisotropic medium can be extracted with a Fourier technique in the form of surface integrals. Also, a detailed discussion focusses on the types of mediums for which these integrals can be explicitly evaluated. The basis of this review on depolarization dyadics is a recent review paper on fields in the source region [4] which may be consulted for further details.

2. Theoretical Background

Time–harmonic electromagnetic fields are being considered and a time–dependence of $\exp(-i\omega t)$ of field quantities is assumed throughout and suppressed (ω is the angular frequency). In 6–vector notation, the complex–valued field and source phasors are defined by[1]

$$\underline{\mathbf{F}}(\underline{x}) = \begin{pmatrix} \underline{E}(\underline{x}) \\ \underline{H}(\underline{x}) \end{pmatrix}, \qquad \underline{\mathbf{Q}}(\underline{x}) = \begin{pmatrix} \underline{J}_e(\underline{x}) \\ \underline{J}_m(\underline{x}) \end{pmatrix}, \tag{1}$$

where $\underline{E}(\underline{x})$ and $\underline{H}(\underline{x})$ are standard electric and magnetic field phasors and $\underline{J}_e(\underline{x})$ and $\underline{J}_m(\underline{x})$ are the electric and magnetic current densities, respectively. The most general linear medium, commonly referred to as a *bianisotropic medium*, is described by the frequency–dependent constitutive relations

$$\begin{pmatrix} \underline{D}(\underline{x}) \\ \underline{B}(\underline{x}) \end{pmatrix} = \underline{\underline{\mathbf{K}}} \bullet \underline{\mathbf{F}}(\underline{x}), \qquad \underline{\underline{\mathbf{K}}} = \begin{pmatrix} \underline{\underline{\epsilon}} & \underline{\underline{\xi}} \\ \underline{\underline{\zeta}} & \underline{\underline{\mu}} \end{pmatrix}, \tag{2}$$

where $\underline{D}(\underline{x})$ is the dielectric displacement and $\underline{B}(\underline{x})$ is the magnetic induction. As for the 3×3 constituent dyadics of $\underline{\underline{\mathbf{K}}}$, $\underline{\underline{\epsilon}}$ and $\underline{\underline{\mu}}$ are the well–known permittivity and permeability dyadics, whereas $\underline{\underline{\xi}}$ and $\underline{\underline{\zeta}}$ are the two magnetoelectric dyadics. The medium parameters will normally be complex scalars or pseudoscalars and it will be assumed that they fulfil the general covariance constraint: Trace $\left(\underline{\underline{\xi}} \bullet \underline{\underline{\mu}}^{-1} + \underline{\underline{\mu}}^{-1} \bullet \underline{\underline{\zeta}}\right) = 0$, [5], dictated by the structure of modern electromagnetic theory.

Consequently, the Maxwell equations can be given in the compact form:

$$\left[\underline{\underline{\mathbf{L}}}(\nabla) + i\omega\underline{\underline{\mathbf{K}}}\right] \bullet \underline{\mathbf{F}}(\underline{x}) = \underline{\mathbf{Q}}(\underline{x}), \qquad \underline{\underline{\mathbf{L}}}(\nabla) = \begin{pmatrix} \underline{\underline{0}} & \nabla \times \underline{\underline{I}} \\ -\nabla \times \underline{\underline{I}} & \underline{\underline{0}} \end{pmatrix}, \tag{3}$$

where $\underline{\underline{I}}$ is the 3×3 unit dyadic. Because the differential operator $\underline{\underline{\mathbf{L}}}(\nabla)$ and the constitutive dyadic $\underline{\underline{\mathbf{K}}}$ are both linear, (3) permits a solution representation of the form

$$\underline{\mathbf{F}}(\underline{x}) = \underline{\mathbf{F}}_h(\underline{x}) + \int_{V'} \underline{\underline{\mathbf{G}}}(\underline{x}, \underline{x}') \bullet \underline{\mathbf{Q}}(\underline{x}')\, d^3\underline{x}', \tag{4}$$

wherein V' is a volume such that $\underline{\mathbf{Q}}(\underline{x}) = \underline{\mathbf{0}}$ for $\underline{x} \notin V'$. The field $\underline{\mathbf{F}}_h(\underline{x})$ is a solution of (3) for $\underline{\mathbf{Q}}(\underline{x}) \equiv \underline{\mathbf{0}}$, i.e., mathematically speaking it is the complementary function. The entity $\underline{\underline{\mathbf{G}}}(\underline{x}, \underline{x}')$ is generally called the *dyadic Green function*, abbreviated DGF henceforth. It contains the standard 3×3 DGFs $\underline{\underline{G}}_{ee}$, $\underline{\underline{G}}_{em}$, $\underline{\underline{G}}_{me}$ and $\underline{\underline{G}}_{mm}$. It now follows from (3) and (4) that the DGF fulfils the *dyadic differential equation*

$$\left[\underline{\underline{\mathbf{L}}}(\nabla) + i\omega\underline{\underline{\mathbf{K}}}\right] \bullet \underline{\underline{\mathbf{G}}}(\underline{x}, \underline{x}') = \delta(\underline{x} - \underline{x}')\, \underline{\underline{\mathbf{I}}}, \tag{5}$$

where $\delta(\underline{x} - \underline{x}')$ is the Dirac delta function and $\underline{\underline{\mathbf{I}}}$ is the unit dyadic.

The field representation (4) holds for all $\underline{x}$. While it can be applied directly in the case $\underline{x} \notin V'$, special care is required when $\underline{x} \in V'$ because of singularities in the integrand. That case is of special interest in this context as it leads to the concept of the depolarization dyadics.

3. Field Representation with the Fikioris Technique

Two broad approaches for dealing with the problem of correctly evaluating the integral in (4) for $\underline{x} \in V'$ have been developed. The first of these was pioneered by Van Bladel [6, 7], and it was

[1] 3–vectors (6–vectors) are in normal (bold) face and underlined whereas 3×3 dyadics (6×6 dyadics) are in normal (bold) face and underlined twice.

later systematically generalized by Yaghjian [8]. Their work and that of others during the 1980s relates to *isotropic dielectric–magnetic mediums* only, i.e., for $\underline{\underline{\epsilon}} = \epsilon\,\underline{\underline{I}}$, $\underline{\underline{\xi}} = \underline{\underline{0}}$, $\underline{\underline{\zeta}} = \underline{\underline{0}}$, $\underline{\underline{\mu}} = \mu\,\underline{\underline{I}}$.

More recent work, based upon the principles of electromagnetostatics, reported some progress in the application of that technique to uniaxial [9] and general, homogeneous bianisotropic mediums [10]. The essence of the Van Bladel/Yaghjian technique is that the integral in (4) is treated by excluding a small convex region (often called *exclusion region* or *exclusion volume*) surrounding the source point of the integration region. The integral can then be evaluated and, eventually, the linear dimensions of the exclusion region are shrunk to zero — a procedure that effectively amounts to *principal value integration*.

An alternative approach was developed by Fikioris [11] in which the volume of the exclusion region is not required to be infinitesimally small in size. As a consequence, estimates for the electromagnetic field in the source region thus obtained differ from those arising from the Van Bladel/Yaghjian technique. In particular, the Fikioris technique gives rise to a dependence on the electrical size of the exclusion region in addition to that on its shape; see [12] for further information. Thus, the Fikioris technique amounts to the *regularization* of a divergent integral [13]. It is also worth noting that the Fikioris technique simplifies to the Van Bladel/Yaghjian technique when the exclusion volume is made infinitesimally small.

While these developments concerning the Fikioris technique also related to isotropic mediums only, recent years have seen significant extensions of the method to more complicated, linear homogeneous mediums. Initial application of this approach for a *bi–isotropic medium* [14] was followed by an extension to a *simple uniaxial bianisotropic medium* (i.e., an affinely transformed bi–isotropic medium) [15]. The Fikioris technique was also applied to an *uniaxial dielectric–magnetic medium* [16], an *axially uniaxial bianisotropic medium* (AUBM) [17], a *simple symmetric bianisotropic medium* [18] and an *affinely transformable AUBM* [19]. Results may be extended to mediums related to these by affine or other field transformations [20].

The essence of the Fikioris technique can be outlined in a straightforward way. In order to transform (4) into a suitable format, a volume V'' is defined in such a way that it constitutes a convex region surrounding $\underline{x} = \underline{x}'$. Also, an auxiliary dyadic Green function $\underline{\underline{\mathbf{G}}}_P(\underline{x},\underline{x}')$, called the *Poisson dyadic*, must be introduced. Hence, (4) may be rewritten as

$$\begin{aligned}\underline{\mathbf{F}}(\underline{x}) = \underline{\mathbf{F}}_h(\underline{x}) + \int_{V'-V''} \underline{\underline{\mathbf{G}}}(\underline{x},\underline{x}') \bullet \underline{\mathbf{Q}}(\underline{x}')\, d^3\underline{x}' + \\ \int_{V''} \left[\underline{\underline{\mathbf{G}}}(\underline{x},\underline{x}') \bullet \underline{\mathbf{Q}}(\underline{x}') - \underline{\underline{\mathbf{G}}}_P(\underline{x},\underline{x}') \bullet \underline{\mathbf{Q}}(\underline{x})\right] d^3\underline{x}' + \underline{\underline{\mathbf{D}}}(\underline{x}) \bullet \underline{\mathbf{Q}}(\underline{x})\,. \qquad (6)\end{aligned}$$

This representation is an exact formula without any simplifications, applicable to a general, linear bianisotropic medium as described by the constitutive dyadic in (2). In connection with the field representation (6) it must be observed that an explicit formula for the infinite–medium DGF $\underline{\underline{\mathbf{G}}}(\underline{x},\underline{x}')$ — and consequently also for the Poisson dyadic $\underline{\underline{\mathbf{G}}}_P(\underline{x},\underline{x}')$ — of a general, linear, homogeneous bianisotropic medium is not available at this time (see [21, 22] for review works on infinite–medium DGFs for complex mediums). This entails then that the dyadic function $\underline{\underline{\mathbf{D}}}(\underline{x})$ in (6), which is called the *depolarization dyadic*, and which is given by

$$\underline{\underline{\mathbf{D}}}(\underline{x}) = \int_{V''} \underline{\underline{\mathbf{G}}}_P(\underline{x},\underline{x}')\, d^3\underline{x}'\,, \qquad (7)$$

can also not be derived for a general, linear bianisotropic medium.

While the foregoing developments have fully defined the depolarization dyadic $\underline{\underline{\mathbf{D}}}(\underline{x})$, it is in approximations of (6) that its role and significance receives further illumination. One may assume, for example, that V' is itself convex in shape and also *electrically small*, i.e., its maximum chord length is smaller than a tenth of the principal wavelengths in the medium, say. Then, as the Fikioris technique does not require V'' to be infinitesimally small, one can set $V' = V''$. As a consequence, the representation of the field simplifies to the extent that the integral over the region $V' - V''$ on the right–hand side of (6) vanishes.

If one makes the additional assumption that the current density distribution $\underline{\mathbf{Q}}(\underline{x})$ is uniform inside $V' = V''$, one obtains the *long–wavelength estimate* from the Fikioris technique as

$$\underline{\mathbf{F}}_{lw}(\underline{x}) \cong \underline{\mathbf{F}}_{h}(\underline{x}) + \left[\underline{\underline{\mathbf{M}}}(\underline{x}) + \underline{\underline{\mathbf{D}}}(\underline{x})\right] \bullet \underline{\mathbf{Q}}(\underline{x})\,, \tag{8}$$

for $\underline{x} \in V'$, where

$$\underline{\underline{\mathbf{M}}}(\underline{x}) = \int_{V'} \left[\underline{\underline{\mathbf{G}}}(\underline{x},\underline{x}') - \underline{\underline{\mathbf{G}}}_{P}(\underline{x},\underline{x}')\right] d^3\underline{x}'\,. \tag{9}$$

If, finally, the term involving $\underline{\underline{\mathbf{M}}}(\underline{x})$ is ignored, the *Rayleigh estimate* of the electromagnetic field in the source region arises in the form

$$\underline{\mathbf{F}}_{Ray}(\underline{x}) \cong \underline{\mathbf{F}}_{h}(\underline{x}) + \underline{\underline{\mathbf{D}}}(\underline{x}) \bullet \underline{\mathbf{Q}}(\underline{x})\,. \tag{10}$$

In this last relation we can now directly recognize the significance of the depolarization dyadic as a mapping from the source $\underline{\mathbf{Q}}(\underline{x})$ to the scattered field $\underline{\mathbf{F}}_{Ray}(\underline{x}) - \underline{\mathbf{F}}_{h}(\underline{x})$. Thus, the complete response of the individual scatterer is seen to be contained in the depolarization dyadic $\underline{\underline{\mathbf{D}}}(\underline{x})$.

4. The Depolarization Dyadic

As far as the depolarization dyadic $\underline{\underline{\mathbf{D}}}(\underline{x})$ is concerned, the situation appears now like this: if, for a specific type of medium the dyadic Green function $\underline{\underline{\mathbf{G}}}(\underline{x},\underline{x}')$ is known, one can obtain the Poisson dyadic $\underline{\underline{\mathbf{G}}}_{P}(\underline{x},\underline{x}')$ and thus finds a representation of the depolarization dyadic in terms of a volume integral by virtue of (7). Due to the structure of $\underline{\underline{\mathbf{G}}}_{P}(\underline{x},\underline{x}')$, that volume integral can always be transformed into a surface integral with the help of Gauss' theorem.

Consider, for example (see [16] for mathematical details), an *uniaxial, dielectric–magnetic* medium, defined by constitutive dyadics

$$\underline{\underline{\epsilon}} = \epsilon_t\,\underline{\underline{I}} + (\epsilon_c - \epsilon_t)\,\underline{c}\,\underline{c}\,, \qquad \underline{\underline{\xi}} = \underline{\underline{0}}\,, \qquad \underline{\underline{\zeta}} = \underline{\underline{0}}\,, \qquad \underline{\underline{\mu}} = \mu_t\,\underline{\underline{I}} + (\mu_c - \mu_t)\,\underline{c}\,\underline{c}\,, \tag{11}$$

where $\underline{c}$ is a unit vector. After convenient decomposition of $\underline{\underline{\mathbf{D}}}(\underline{x})$ according to

$$\underline{\underline{\mathbf{D}}}(\underline{x}) = \begin{pmatrix} \underline{\underline{D}}_{ee}(\underline{x}) & \underline{\underline{D}}_{em}(\underline{x}) \\ \underline{\underline{D}}_{me}(\underline{x}) & \underline{\underline{D}}_{mm}(\underline{x}) \end{pmatrix}, \tag{12}$$

into 3×3 component dyadics, we find

$$\underline{\underline{D}}_{ee} = \frac{1}{i\omega}\, q_e\, \underline{\underline{L}}_{e} \bullet \underline{\underline{\epsilon}}^{-1}\,, \qquad \underline{\underline{D}}_{em} = \underline{\underline{0}}\,, \qquad \underline{\underline{D}}_{me} = \underline{\underline{0}}\,, \qquad \underline{\underline{D}}_{mm} = \frac{1}{i\omega}\, q_m\, \underline{\underline{L}}_{m} \bullet \underline{\underline{\mu}}^{-1}\,, \tag{13}$$

where $q_e = \epsilon_c/\epsilon_t$ and $q_m = \mu_c/\mu_t$. The dyadics $\underline{\underline{L}}_{e,m}$ can be explicitly evaluated as

$$\underline{\underline{L}}_{e,m} = \frac{1}{2}\left(\frac{1}{q_{e,m}} + \Lambda(q_{e,m})\right)\left(\underline{\underline{I}} - \underline{c}\,\underline{c}\right) - \Lambda(q_{e,m})\underline{c}\,\underline{c}\,, \tag{14}$$

$$\Lambda(q) = \frac{1}{1-q}\left(1 - \frac{1}{(q-1)^{1/2}}\,\tan^{-1}\sqrt{q-1}\right). \tag{15}$$

However, as an alternative to the route taken thus far, it turns out that $\underline{\underline{\mathbf{D}}}(\underline{x})$ may be obtained independently through a Fourier space analysis which does not require a complete knowledge of the infinite–medium DGF of the medium. The procedure was first implemented for a general *anisotropic dielectric medium* [23] (for a spherical exclusion volume) and then generalized to the most general, linear, homogeneous bianisotropic medium with constitutive dyadic as given by (2) (for an ellipsoidal exclusion volume) [24]. Following [24], let the surface of an ellipsoidal region V' be parametrized as $\underline{x}_s(\theta,\phi) = \delta\,\underline{\underline{U}} \bullet \hat{\underline{x}}(\theta,\phi)$ where $\hat{\underline{x}}(\theta,\phi)$ is the radial unit vector in a spherical coordinate system (with angular coordinates θ and ϕ) located at the centre of V'; δ is a linear measure of the size of V', and $\underline{\underline{U}}$ is the symmetric shape dyadic ($\underline{\underline{U}} = \underline{\underline{I}}$ when V' is spherical).

Then, the 3×3 depolarization dyadics in (12) are given by

$$\underline{\underline{D}}_{\lambda\lambda'} = \underline{\underline{U}}^{-1} \bullet \tilde{\underline{\underline{D}}}_{\lambda\lambda'} \bullet \underline{\underline{U}}^{-1}\,, \quad (\lambda,\lambda' = e,m)\,. \tag{16}$$

The 3×3 dyadic $\tilde{\underline{\underline{D}}}_{\lambda\lambda'}$ is calculated as the double integral

$$\tilde{\underline{\underline{D}}}_{\lambda\lambda'} = \frac{1}{4\pi i\omega}\int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi} d\theta\,\sin\theta \frac{(\hat{\underline{x}} \bullet \underline{\underline{\tau}}_{\lambda\lambda'} \bullet \hat{\underline{x}})\,\hat{\underline{x}}\,\hat{\underline{x}}}{(\hat{\underline{x}} \bullet \underline{\underline{\epsilon}}' \bullet \hat{\underline{x}})(\hat{\underline{x}} \bullet \underline{\underline{\mu}}' \bullet \hat{\underline{x}}) - (\hat{\underline{x}} \bullet \underline{\underline{\zeta}}' \bullet \hat{\underline{x}})(\hat{\underline{x}} \bullet \underline{\underline{\xi}}' \bullet \hat{\underline{x}})}\,, \tag{17}$$

$$\underline{\underline{\epsilon}}' = \underline{\underline{U}}^{-1} \bullet \underline{\underline{\epsilon}} \bullet \underline{\underline{U}}^{-1}\,, \quad \underline{\underline{\xi}}' = \underline{\underline{U}}^{-1} \bullet \underline{\underline{\xi}} \bullet \underline{\underline{U}}^{-1}\,, \quad \underline{\underline{\zeta}}' = \underline{\underline{U}}^{-1} \bullet \underline{\underline{\zeta}} \bullet \underline{\underline{U}}^{-1}\,, \quad \underline{\underline{\mu}}' = \underline{\underline{U}}^{-1} \bullet \underline{\underline{\mu}} \bullet \underline{\underline{U}}^{-1}\,, \tag{18}$$

$$\underline{\underline{\tau}}_{ee} = \underline{\underline{\mu}}'\,, \quad \underline{\underline{\tau}}_{em} = -\underline{\underline{\xi}}'\,, \quad \underline{\underline{\tau}}_{me} = -\underline{\underline{\zeta}}'\,, \quad \underline{\underline{\tau}}_{mm} = \underline{\underline{\epsilon}}'\,. \tag{19}$$

It is clear that in general the double integral in (17) needs to be evaluated numerically. Explicit, closed–form evaluations (for ellipsoidal, spherical, cylindrical and cubical exclusion volumes) exist for the isotropic case: both for achiral isotropic mediums [8] as well as for chiral mediums [14]. For uniaxial dielectric–magnetic and for uniaxial bianisotropic mediums where at least one of the medium dyadics $\underline{\underline{\epsilon}}$, $\underline{\underline{\xi}}$, $\underline{\underline{\zeta}}$, $\underline{\underline{\mu}}$ is of the form $\underline{\underline{a}} = a_c\,\underline{\underline{I}} + a_c\,\underline{c}\,\underline{c}$ ($\underline{c}$ is again a unit vector) explicit expressions (in terms of inverse trigonometric functions) are also available [23], [16], [25]; [17], [26]. The most recently obtained results in closed form relate to a *biaxial dielectric anisotropic medium* with $\underline{\underline{\epsilon}} = \epsilon_x\,\underline{u}_x\,\underline{u}_x + \epsilon_y\,\underline{u}_y\,\underline{u}_y + \epsilon_z\,\underline{u}_z\,\underline{u}_z$, $\underline{\underline{\xi}} = \underline{\underline{0}}$, $\underline{\underline{\zeta}} = \underline{\underline{0}}$, $\underline{\underline{\mu}} = \mu\,\underline{\underline{I}}$, where $\underline{u}_x$, $\underline{u}_y$, $\underline{u}_z$ are the unit vectors of a cartesian coordinate system. The depolarization dyadic $\underline{\underline{\mathbf{D}}}(\underline{x})$ is given in terms of elliptic functions of the first and second kind [27]. Finally, the closed–form results obtained for isotropic/uniaxial/biaxial mediums can be trivially extended to mediums which contain arbitrary skew–symmetric elements in their 3×3 constitutive dyadics, see [24], [27], because any skew–symmetry is filtered out in the integral representation (17).

5. Concluding Remarks

It was shown that depolarization dyadics play an important role in electromagnetic field representations. They arise through regularization procedures which guarantee that the field representations also remain valid when the field point is in the source region. Their significance can be interpreted in a straightforward way, namely that within certain well–defined approximations, particularly the Rayleigh estimate, they provide the complete information about the electromagnetic response of an individual scatterer.

The depolarization dyadics of a general, linear bianisotropic medium can be represented in terms of a two–dimensional integral. There are many special cases for which the integrations can be performed explicitly to yield closed–form expressions. In general, however, these integrals lend themselves easily to numerical treatment that require only small computational resources. Such a numerical approach is also economic because in their most important application, depolarization dyadics are used in homogenization calculations that are necessarily numerical by nature.

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