Hertz Potentials in Complex Medium Electromagnetics

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Abstract

A brief outline of the mathematical technique of scalar Hertz potentials is provided. It is shown that the technique can be successfully implemented if all four constitutive dyadics of a linear bianisotropic medium have a gyrotropic structure. A medium of that nature is a so–called Faraday chiral medium, and it may even be nonhomogeneous but of a restricted form.

1. Introduction

Approaches to solve electromagnetic field problems can be roughly divided into two categories: *direct* and *indirect*. While the former deal directly with the Maxwell equations for the electromagnetic field vectors, the latter aim to derive, and consequently solve, alternative differential equations that are fulfilled by certain functions — scalar, vector or dyadic. The field vectors themselves then follow from these functions through (relatively simple) manipulations such as differentiations and integrations.

Two of the most prominent indirect solution techniques are (i) Green function methods and (ii) potential methods. In this contribution we shall focus on potentials, and specifically on scalar Hertz potentials and their role in the solution of electromagnetic field problems in complex mediums.

Vector, scalar and Hertz potentials, in their application to (achiral) isotropic mediums, have been prominent mathematical tools since the early stages of the development of the Maxwellian formalism — a fact easily betrayed by their inclusion in the vast majority of standard textbooks on electromagnetic theory. But only in the last two decades has it been possible to extend the concept of scalar Hertz potentials into the realm of more complex mediums, which may be anisotropic, bianisotropic, nonhomogeneous, in their nature.

Here, an outline of the scalar Hertz potential method is given, necessarily short due to the limited available space. Full details including a comprehensive listing of references of the relevant research literature can be found in a recent book chapter [1].

2. Analysis

The Maxwell equations for a general, linear bianisotropic medium are[1]

$$i\omega \left[\underline{\underline{\epsilon}} \bullet \mathbf{E}(\mathbf{x}) + \underline{\underline{\xi}} \bullet \mathbf{H}(\mathbf{x}) \right] + \nabla \times \mathbf{H}(\mathbf{x}) = \mathbf{J}_e(\mathbf{x})\,, \tag{1}$$

$$\nabla \times \mathbf{E}(\mathbf{x}) - i\omega \left[\underline{\underline{\zeta}} \bullet \mathbf{E}(\mathbf{x}) + \underline{\underline{\mu}} \bullet \mathbf{H}(\mathbf{x}) \right] = -\mathbf{J}_m(\mathbf{x})\,. \tag{2}$$

[1]Vectors are in bold face whereas dyadics are in normal face and underlined twice. Contraction of indices is symbolized by a dot and $\mathbf{a}\,\mathbf{b}$ is a dyadic product; the unit dyadic is $\underline{\underline{I}}$.

$\mathbf{E}(\mathbf{x})$ and $\mathbf{H}(\mathbf{x})$ are the electric and magnetic field vectors, whereas $\mathbf{J}_e(\mathbf{x})$ and $\mathbf{J}_m(\mathbf{x})$ are the prescribed electric and magnetic current densities (a time–dependence of $\exp(-i\omega t)$ is implicit). At this stage no restrictions are imposed on the specific form of the four constitutive dyadics in (1), (2).

To develop a field representation in terms of scalar Hertz potentials, a partial scalarization of the differential equations (1), (2), is carried out with respect to an arbitrarily chosen direction specified by a unit vector $\mathbf{c}$ — once again, full mathematical details can be found in [1]. The decompositions

$$\mathbf{E} = \mathbf{E}_t + E_c\,\mathbf{c}\,, \qquad \mathbf{H} = \mathbf{H}_t + H_c\,\mathbf{c}\,, \tag{3}$$

are introduced, whereby $\mathbf{E}_t \cdot \mathbf{c} = 0$ and $\mathbf{H}_t \cdot \mathbf{c} = 0$; $\mathbf{E}_t$ and $\mathbf{H}_t$ being called the *transverse*, E_c and H_c the *longitudinal* components of the field vectors. Similarly, for the electric and magnetic current density:

$$\mathbf{J}_e = \mathbf{J}_{et} + J_{ec}\,\mathbf{c}\,, \qquad \mathbf{J}_m = \mathbf{J}_{mt} + J_{mc}\,\mathbf{c}\,, \tag{4}$$

and also for the derivative operations:

$$\nabla = \nabla_t + \mathbf{c}\,\frac{\partial}{\partial x_c}\,, \qquad \nabla^2 = \nabla_t^2 + \frac{\partial^2}{\partial x_c^2}\,, \tag{5}$$

wherein ∇^2 is the Laplace operator.

The $\mathbf{c}$–components of (1), (2) provide two scalar expression for the components E_c and H_c and one finds, after some algebra,

$$E_c = \mathbf{P}_{ee} \cdot \mathbf{E}_t + \mathbf{P}_{eh} \cdot \mathbf{H}_t + K_e\,, \tag{6}$$

$$H_c = \mathbf{P}_{he} \cdot \mathbf{E}_t + \mathbf{P}_{hh} \cdot \mathbf{H}_t + K_h\,, \tag{7}$$

where $\mathbf{P}_{ee}$, $\mathbf{P}_{eh}$, $\mathbf{P}_{he}$ and $\mathbf{P}_{hh}$ are 2×2 dyadic operators while K_e and K_h are source terms. Now, a system for the transverse components $\mathbf{E}_t$ and $\mathbf{H}_t$ is derived in the form

$$-\frac{\partial \mathbf{E}_t}{\partial x_c} + i\omega\,\underline{\underline{M}}_{ee} \cdot \mathbf{E}_t + i\omega\,\underline{\underline{M}}_{eh} \cdot \mathbf{H}_t = \mathbf{q}_{te}\,, \tag{8}$$

$$-\frac{\partial \mathbf{H}_t}{\partial x_c} + i\omega\,\underline{\underline{M}}_{he} \cdot \mathbf{E}_t + i\omega\,\underline{\underline{M}}_{hh} \cdot \mathbf{H}_t = \mathbf{q}_{th}\,. \tag{9}$$

where $\underline{\underline{M}}_{ee}$, $\underline{\underline{M}}_{eh}$, $\underline{\underline{M}}_{he}$ and $\underline{\underline{M}}_{hh}$ are 2×2 dyadic differential operators of second order and $\mathbf{q}_{te}$ and $\mathbf{q}_{th}$ are 2–vector source terms.

The crucial step is now the introduction of scalar potentials through

$$\mathbf{E}_t = \nabla_t\,\Phi + \nabla_t \times \Theta\,\mathbf{c}\,, \tag{10}$$

$$\mathbf{H}_t = \nabla_t\,\Pi + \nabla_t \times \Psi\,\mathbf{c}\,. \tag{11}$$

Straightforward manipulation of (6), (7), then leads to formulas for the longitudinal components E_c and H_c, in symbolic form:

$$E_c = E_c\,(\Phi\,, \Theta\,, \Pi\,, \Psi\,; J_{ec}\,, J_{mc})\;, \tag{12}$$

$$H_c = H_c\,(\Phi\,, \Theta\,, \Pi\,, \Psi\,; J_{ec}\,, J_{mc})\;. \tag{13}$$

The expressions (10)–(13) constitute a representation of all field components in terms of the four scalar functions Φ, Θ, Π and Ψ; substitution into the available differential equations produces a system of four differential equations for the four scalar potentials. At that stage no direct gain has been achieved as one has simply exchanged solving a system of partial differential equations of second order for $\mathbf{E}_t$ and $\mathbf{H}_t$ with solving a corresponding system for the four scalar potentials.

The question is therefore: can two of the four scalar potentials be eliminated in a simple manner so that the field vectors **E** and **H** can be represented *in full* by two scalar Hertz potentials plus, perhaps, some auxiliary functions that are required to deal with specific source terms?

The answer to that question is that such an elimination is not possible for the general, linear bianisotropic medium characterized by $\underline{\underline{\epsilon}}$, $\underline{\underline{\xi}}$, $\underline{\underline{\zeta}}$ and $\underline{\underline{\mu}}$. Conditions on the structure of these constitutive dyadics must be imposed and the desired reduction is possible only if the constitutive dyadics are of the form

$$\begin{aligned} \underline{\underline{\sigma}} &= \sigma_t \underline{\underline{I}} + (\sigma_c - \sigma_t)\, \mathbf{c}\,\mathbf{c} - i\sigma_g\, \mathbf{c} \times \underline{\underline{I}} \\ &= \begin{pmatrix} \sigma_t & i\sigma_g & 0 \\ -i\sigma_g & \sigma_t & 0 \\ 0 & 0 & \sigma_c \end{pmatrix}, \qquad \underline{\underline{\sigma}} = \underline{\underline{\epsilon}}\,, \underline{\underline{\xi}}\,, \underline{\underline{\zeta}}\,, \underline{\underline{\mu}}\,. \end{aligned} \tag{14}$$

All four constitutive dyadics are of gyrotropic form, and a medium of that nature is called a Faraday chiral medium [2]. In the dyadics (14), σ_t and σ_c characterize the uniaxiality of the medium whereas σ_g is responsible for the gyrotropic or gyrotropic–like nature. The scalar Hertz potential formalism for homogeneous Faraday chiral mediums has been outlined in detail elsewhere [2]. The field representation, for all the types of mediums for which the scalar Hertz potential technique is then applicable, can symbolically be given by

$$\mathbf{E} = \mathbf{E}\,(\Theta\,, \Psi\,; J_{ec}\,, u_e\,, v_e\,, J_{mc}\,, u_m\,, v_m)\ , \tag{15}$$

$$\mathbf{H} = \mathbf{H}\,(\Theta\,, \Psi\,; J_{ec}\,, u_e\,, v_e\,, J_{mc}\,, u_m\,, v_m)\ , \tag{16}$$

because usually the two potentials that get eliminated are Φ and Π.

Typically, the two functions Θ and Ψ — the scalar Hertz potentials — fulfil a system of differential equations of the structure

$$L_1\Theta + L_2\Psi = s_1\,, \tag{17}$$

$$L_3\Theta + L_4\Psi = s_2\,, \tag{18}$$

where the exact form of the second order partial differential operators L_1, L_2, L_3 and L_4 depends on the specific type of bianisotropic medium being analysed; s_1 and s_2 are source terms. In general, thus, Θ and Ψ are coupled but decoupling occurs if the medium becomes sufficiently simple (as happens, for example, for a uniaxial dielectric medium [3]).

Finally, what is the significance of the functions u_e, u_m, v_e and v_m that have entered the field representation (15), (16)? These are *auxiliary* functions defined by

$$\mathbf{J}_{et} = \nabla_t\, u_e + \nabla_t \times\, v_e\, \mathbf{c}\,, \tag{19}$$

$$\mathbf{J}_{mt} = \nabla_t\, u_m + \nabla_t \times\, v_m\, \mathbf{c}\,. \tag{20}$$

Consequently, the four auxiliary functions are calculated from the differential equations

$$\nabla_t^2 u_p = \nabla_t \bullet \mathbf{J}_{pt}\,, \quad \nabla_t^2 v_p = \nabla_t \bullet\ (\mathbf{c} \times \mathbf{J}_{pt})\ , \quad p = e, m\,, \tag{21}$$

and it is immediately clear from these expressions that they have nothing to do with the medium as such but are simply generated by the mathematical formalism to accommodate transverse current densities.

3. Discussion

The brief outline of the scalar Hertz potential technique is concluded here with a number of observations; see [1] for more detailed discussions of the individual points.

- Not mentioned previously is the possible generalization to nonhomogeneous mediums, where the nonhomogeneity is of a restricted kind. Namely, all constitutive parameters can depend on $x_c (= \mathbf{x} \bullet \mathbf{c})$, the coordinate parallel to the distinguished axis $\mathbf{c}$. Thus, the most general type of medium that is amenable to the scalar Hertz potential technique is characterized by constitutive dyadics of the form

$$\underline{\underline{\epsilon}}(x_c) = \epsilon_t(x_c)\,\underline{\underline{I}} + \Big[\epsilon_c(x_c) - \epsilon_t(x_c)\Big]\mathbf{c}\,\mathbf{c} - i\epsilon_g(x_c)\,\mathbf{c}\times\underline{\underline{I}}\,, \tag{22}$$

$$\underline{\underline{\xi}}(x_c) = \xi_t(x_c)\,\underline{\underline{I}} + \Big[\xi_c(x_c) - \xi_t(x_c)\Big]\mathbf{c}\,\mathbf{c} - i\xi_g(x_c)\,\mathbf{c}\times\underline{\underline{I}}\,, \tag{23}$$

$$\underline{\underline{\zeta}}(x_c) = \zeta_t(x_c)\,\underline{\underline{I}} + \Big[\zeta_c(x_c) - \zeta_t(x_c)\Big]\mathbf{c}\,\mathbf{c} - i\zeta_g(x_c)\,\mathbf{c}\times\underline{\underline{I}}\,, \tag{24}$$

$$\underline{\underline{\mu}}(x_c) = \mu_t(x_c)\,\underline{\underline{I}} + \Big[\mu_c(x_c) - \mu_t(x_c)\Big]\mathbf{c}\,\mathbf{c} - i\mu_g(x_c)\,\mathbf{c}\times\underline{\underline{I}}\,. \tag{25}$$

  All constitutive dyadics have a gyrotropic structure and are spatially nonhomogeneous through the dependence on x_c and are thus nonhomogeneous Faraday chiral mediums [2].

- The auxiliary functions u_e, u_m, v_e and v_m are determined by (19)–(21); it is noted that

$$\mathbf{J}_{et} = \mathbf{0} \quad \Longrightarrow \quad u_e = 0\,,\ v_e = 0\,, \tag{26}$$

$$\mathbf{J}_{mt} = \mathbf{0} \quad \Longrightarrow \quad u_m = 0\,,\ v_m = 0\,. \tag{27}$$

  Consequently, they can be completely omitted from the mathematical formalism if the medium is free of sources, i.e., $\mathbf{J}_e = \mathbf{0}$, $\mathbf{J}_m = \mathbf{0}$, or if the current densities are purely longitudinal, i.e., $\mathbf{J}_e = J_{ec}\mathbf{c}$, $\mathbf{J}_m = J_{mc}\mathbf{c}$.

- With the fundamental system of differential equations (that for the scalar Hertz potentials) that needs to be solved being a scalar problem, Green functions that may be introduced subsequently can be of the simpler scalar rather than much more complicated dyadic nature.

- The introduction of the scalar Hertz potentials is not a unique process. New scalar Hertz potentials $\Theta^{new}(\mathbf{x})$ and $\Psi^{new}(\mathbf{x})$ may be defined via

$$\Theta^{new}(\mathbf{x}) = \gamma_1\,\Theta(\mathbf{x}) + \gamma_2\,\Psi(\mathbf{x})\,, \qquad \Psi^{new}(\mathbf{x}) = \gamma_3\,\Theta(\mathbf{x}) + \gamma_4\,\Psi(\mathbf{x})\,, \tag{28}$$

  where γ_n, $n = 1,2,3,4$, are arbitrary complex constants (provided $\gamma_1\gamma_4 \neq \gamma_2\gamma_3$).

In conclusion, the scalar Hertz potential technique provides a very successful mathematical method to deal with electromagnetic field problems in complex mediums.

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References

[1] W.S. Weiglhofer, "Scalar Hertz potentials for linear bianisotropic mediums," in O.N. Singh and A. Lakhtakia (eds.), *Electromagnetic Fields in Unconventional Structures and Materials*. New York: Wiley, 2000, 1–37.

[2] W.S. Weiglhofer and S.O. Hansen, "Faraday chiral media revisited–I: Fields and sources," *IEEE Trans Antennas Propagat* **47**, 807–814, 1999.

[3] W.S. Weiglhofer, "Scalar Hertz potentials for nonhomogeneous, uniaxial dielectric–magnetic mediums," *Department of Mathematics, University of Glasgow*, preprint 2000/08, March 2000.