Waves and Fields: From Uniaxial to Biaxial Mediums, in Between and Beyond

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Abstract

First, a consistent perspective for the formulation of constitutive dyadics for biaxial mediums — for the anisotropic dielectric and the full bianisotropic cases — is provided. Then, the connection between the existence of closed–form, infinite–medium, dyadic Green functions and the factorization properties of certain scalar differential operators is explored by focusing on a special type of homogeneous, anisotropic, dielectric medium. Its anisotropy is of a higher degree of complexity than an uniaxial medium's but falls short of a fully biaxial medium's.

1. From Uniaxiality to Biaxiality

An anisotropic dielectric medium of the simplest type has a relative permittivity dyadic $\underline{\underline{\epsilon}}$ that is *uniaxial*. Stated as[1]

$$\underline{\underline{\epsilon}}_{uni} = \epsilon_a \,\underline{\underline{I}} + \epsilon_b \,\mathbf{u}\,\mathbf{u}\,, \tag{1}$$

it employs two complex–valued scalars (i.e., ϵ_a and ϵ_b) and one unit vector (i.e., $\mathbf{u}$) which is parallel to the sole distinguished axis of the medium [1]. Extending the structure of (1) to the permeability dyadic and the magnetoelectric dyadics as well, we arrive at a *uniaxial bianisotropic* medium [2].

Generalization from medium uniaxiality to *biaxiality* requires the introduction of a second distinguished axis. Recently, we [3] put forward a consistent approach to that issue for *biaxial bianisotropic mediums*, delineating their frequency–dependent constitutive relations as

$$\mathbf{D}(\mathbf{x}) = \epsilon_0 \,\underline{\underline{\epsilon}}_{bi} \bullet \mathbf{E}(\mathbf{x}) + (\sqrt{\epsilon_0/\mu_0})\,\underline{\underline{\alpha}}_{bi} \bullet \mathbf{B}(\mathbf{x})\,, \tag{2}$$

$$\mathbf{H}(\mathbf{x}) = (\sqrt{\epsilon_0/\mu_0})\,\underline{\underline{\beta}}_{bi} \bullet \mathbf{E}(\mathbf{x}) + (1/\mu_0)\,\underline{\underline{\chi}}_{bi} \bullet \mathbf{B}(\mathbf{x})\,. \tag{3}$$

Here, the four constitutive dyadics are given by

$$\underline{\underline{\epsilon}}_{bi} = \epsilon_a \,\underline{\underline{I}} + \epsilon_b \,(\mathbf{u}_m \,\mathbf{u}_n + \mathbf{u}_n \,\mathbf{u}_m)\,, \tag{4}$$

$$\underline{\underline{\alpha}}_{bi} = \alpha_a \,\underline{\underline{I}} + \alpha_b \,(\mathbf{u}_m \,\mathbf{u}_n + \mathbf{u}_n \,\mathbf{u}_m)\,, \tag{5}$$

[1] In this paper, vectors are in boldface; dyadics are underlined twice; $\mathbf{a} \bullet \mathbf{b} = \sum_i a_i b_i$; $\underline{\underline{A}} = \mathbf{a}\,\mathbf{b}$ is a dyad with elements $A_{ij} = a_i\, b_j$; $\underline{\underline{I}}$ is the unit dyadic and $\underline{\underline{0}}$ is the null dyadic; the superscript $^{-1}$ indicates inversion of dyadics and differential operators; while $\mathbf{x}$ and $\mathbf{x}'$ are the observation and the source points, respectively.

$$\underline{\underline{\beta}}_{bi} = \beta_a \, \underline{\underline{I}} + \beta_b \, (\mathbf{u}_m \, \mathbf{u}_n + \mathbf{u}_n \, \mathbf{u}_m) \, , \tag{6}$$

$$\underline{\underline{\chi}}_{bi} = \chi_a \, \underline{\underline{I}} + \chi_b \, (\mathbf{u}_m \, \mathbf{u}_n + \mathbf{u}_n \, \mathbf{u}_m) \, , \tag{7}$$

with $\mathbf{u}_m$ and $\mathbf{u}_n$ as two unit vectors that are, in general, neither parallel nor anti–parallel to each other. The simplification $\mathbf{u}_m = \pm\mathbf{u}_n$ leads us back to uniaxial bianisotropic mediums.

The two unit vectors in (4)–(7), and both corresponding distinguished axes [1], are common to all four constitutive dyadics. Furthermore, these dyadics contain eight complex–valued parameters: ϵ_a, ϵ_b, α_a, α_b, β_a, β_b, χ_a and χ_b, while the angle $\phi = \cos^{-1}(\mathbf{u}_m \bullet \mathbf{u}_n)$ is real–valued. Therefore, any biaxial bianisotropic medium is characterized by $8 \times 2 + 1 = 17$ real–valued constitutive scalars. A redundancy in this scheme is filtered out by the *Post constraint* [4]:

$$\text{Trace}\,\left(\underline{\underline{\alpha}}_{bi} - \underline{\underline{\beta}}_{bi}\right) = 0 \quad \Rightarrow \quad 3\,(\alpha_a - \beta_a) + 2\,(\alpha_b - \beta_b)\cos\phi = 0\,. \tag{8}$$

Hence, (4)–(7) actually contain just 15 real–valued independent constitutive scalars. Furthermore, as presently defined, the biaxial bianisotropic medium is a *nonreciprocal* medium.

A *biaxial anisotropic* medium [1] arises as a special case when $\underline{\underline{\alpha}}_{bi} = \underline{\underline{\beta}}_{bi} \equiv \underline{\underline{0}}$ in (4)–(7). The biaxiality is thus purely in the dielectric–magnetic properties, and is described by 4 complex–valued constitutive parameters plus the real–valued angle ϕ. As the Post constraint (8) is then trivially fulfilled, a biaxial anisotropic medium is uniquely described by 9 real–valued constitutive scalars.

Perhaps the most attractive feature of the representation (4)–(7) is found in the possibility that one and the same orthogonal transformation is able to diagonalize all four constitutive dyadics. Full details about that feature, as well as a comprehensive study of electromagnetic wave propagation in biaxial bianisotropic mediums, are available elsewhere [3].

2. Dyadic Green Functions

The dyadic differential equation

$$\underline{\underline{L}}(\nabla) \bullet \underline{\underline{G}}(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}')\,\underline{\underline{I}}\,, \tag{9}$$

constitutes the standard field problem for the electric field phasor, with the $\exp(-i\omega t)$ time–dependence implicit throughout. The dyadic Green function (specifically, of the electric type) is denoted by $\underline{\underline{G}}(\mathbf{x}, \mathbf{x}')$, while $\delta(\mathbf{x} - \mathbf{x}')$ is the Dirac delta function. The specific form of the dyadic, second–order, differential operator $\underline{\underline{L}}(\nabla)$ depends on the type of medium being considered; see Ref. [5] for the relevant exposition of the general Green function technique.

For brevity's sake, we limit our attention here to *anisotropic dielectric* mediums, whose permeability dyadic equals $\underline{\underline{I}}$ and whose magnetoelectric dyadics equal $\underline{\underline{0}}$. A closed–form expression[2] for $\underline{\underline{G}}(\mathbf{x}, \mathbf{x}')$ for an uniaxial dielectric medium is available in textbooks [1], but none is known to exist for a biaxial dielectric medium. The latter is not surprising in view of some results pertaining to the so–called determinant operator of $\underline{\underline{L}}(\nabla)$.

An important step towards a closed–form solution is to determine whether the determinant operator

$$H_{det} = \underline{\underline{L}}(\nabla) \bullet \underline{\underline{L}}_{adj}(\nabla) = \underline{\underline{L}}_{adj}(\nabla) \bullet \underline{\underline{L}}(\nabla)\,, \tag{10}$$

can be factorized into a product of two second–order operators, the *adjoint* operator $\underline{\underline{L}}_{adj}(\nabla)$ in (10) being always of the fourth order by virtue of the structure of the Maxwell equations. It was

[2] The dyadic Green function should be expressible through simple mathematical functions which will most often be *scalar* Green functions of second–order Helmholtzian operators and derivatives as well as linear combinations thereof. It does not include representations in terms of integrals, because such representations can be trivially achieved with spatial Fourier transformations due to the linearity of (9).

first stated in Ref. [6] — see also Ref. [7] — that the determinant operator can be factorized only if the relative permittivity dyadic has the structure

$$\underline{\underline{\epsilon}}_{fact} = \lambda \underline{\underline{I}} + \mathbf{a}\mathbf{b}\,, \tag{11}$$

where λ is a scalar while $\mathbf{a}$ and $\mathbf{b}$ are vectors. Parenthetically, the formulas given in Refs. [6, 7] pertain to the more general anisotropic dielectric–magnetic mediums.

The form of $\underline{\underline{\epsilon}}_{fact}$ stated above is only *sufficient* but not necessary for factorization. Furthermore, the relation between factorization and the availability of closed–form solutions is not clear. In fact, closed–form expressions for dyadic Green functions could not be obtained, despite factorization, for certain types of *uniaxial bianisotropic* mediums, as was first observed in Ref. [8] and explored further in Refs. [9, 2].

3. More than Uniaxial — Not quite Biaxial

We now consider an anisotropic dielectric medium whose relative permittivity dyadic is

$$\underline{\underline{\epsilon}} = \epsilon_a \underline{\underline{I}} + \epsilon_b\, \mathbf{u}_m\, \mathbf{u}_n\,, \tag{12}$$

where $\mathbf{u}_m$ and $\mathbf{u}_n$ are distinct unit vectors. The right side of (12) is clearly equivalent in form to $\underline{\underline{\epsilon}}_{fact}$. In general, the chosen medium is not *reciprocal* as $\underline{\underline{\epsilon}}$ does not equal its transposed dyadic. The uniaxial medium defined through (1) appears as a specialization of (12) when $\mathbf{u}_m = \mathbf{u}_n$. However, (12) also has a connection to a biaxial structure: Upon decomposition of its right side into symmetric and skew–symmetric parts, (12) can be rewritten as

$$\underline{\underline{\epsilon}} = \epsilon_a \underline{\underline{I}} + \frac{\epsilon_b}{2}\,(\mathbf{u}_m\,\mathbf{u}_n + \mathbf{u}_n\,\mathbf{u}_m) + \frac{\epsilon_b}{2}\,(\mathbf{u}_m\,\mathbf{u}_n - \mathbf{u}_n\,\mathbf{u}_m)\,. \tag{13}$$

The first two terms on the right side of (13) have the exact biaxial structure of (4), whereas the last term has a typical *gyrotropic* form. Nevertheless, neither the biaxial dielectric nor the gyrotropic dielectric medium can be obtained from (13) as special cases — because the last two terms on the right side of (13) are far too intimately linked to allow those specializations.

For the medium characterized by (12), we obtain [10]

$$\underline{\underline{L}}_{adj}(\nabla) = H_m \underline{\underline{L}}_e(\nabla) - k^2 \tau\,(\nabla \times \mathbf{u}_n)\,(\nabla \times \mathbf{u}_m)\,, \tag{14}$$

$$H_{det} = -k^2\,(1 + \tau\,\mathbf{u}_m \bullet \mathbf{u}_n)\,H_e\,H_m\,, \tag{15}$$

where $k^2 = \omega^2 \epsilon_0 \epsilon_a \mu_0$, the ratio $\tau = \epsilon_b/\epsilon_a$, and the dyadic operator

$$\underline{\underline{L}}_e(\nabla) = \nabla\,\nabla + k^2\,(1 + \tau\,\mathbf{u}_m \bullet \mathbf{u}_n)\left(\underline{\underline{I}} - \frac{\tau}{1 + \tau\,\mathbf{u}_m \bullet \mathbf{u}_n}\,\mathbf{u}_m\,\mathbf{u}_n\right). \tag{16}$$

Of the two scalar, second–order operators appearing in (14) and (15), $H_m = \nabla^2 + k^2$ is a standard Helmholtz operator due to the magnetic isotropy of the medium, but

$$H_e = \nabla^2 - \frac{\tau}{1 + \tau\,\mathbf{u}_m \bullet \mathbf{u}_n}\,(\nabla \times \mathbf{u}_m) \bullet (\nabla \times \mathbf{u}_n) + k^2\,, \tag{17}$$

is only a Helmholtzian operator as it reflects the dielectric anisotropy of the medium. As anticipated for the chosen medium, it is apparent from (15) that H_{det} indeed factorizes as a product of two second–order operators [10].

Further manipulations then lead to the complete representation of $\underline{\underline{G}}(\mathbf{x},\mathbf{x}')$ in the form

$$\underline{\underline{G}}(\mathbf{x},\mathbf{x}') = \frac{1}{k^2\,(1 + \tau\,\mathbf{u}_m \bullet \mathbf{u}_n)}\left[\underline{\underline{L}}_e(\nabla) g_e(\mathbf{x}\,,\mathbf{x}') + k^2 \tau\,\underline{\underline{M}}(\mathbf{x},\mathbf{x}')\right], \tag{18}$$

where the scalar Green function

$$g_e(\mathbf{x}\,,\mathbf{x}') = \frac{1}{\sqrt{a_x a_y a_z}}\,\frac{\exp\left[ikD(\mathbf{x},\mathbf{x}')\right]}{4\pi D(\mathbf{x},\mathbf{x}')}\,, \tag{19}$$

satisfies the differential equation $H_e\, g_e(\mathbf{x}\,,\mathbf{x}') = -\delta(\mathbf{x}-\mathbf{x}')$, and the structure of the modified distance function $D(\mathbf{x},\mathbf{x}')$ (which also contains $a_{x,y,z}$) is detailed in Ref. [10].

What remains unknown in (18) is the dyadic function $\underline{\underline{M}}(\mathbf{x},\mathbf{x}')$, for which the differential equation

$$H_e H_m\, \underline{\underline{M}}(\mathbf{x},\mathbf{x}') = (\nabla\times\mathbf{u}_n)\ (\nabla\times\mathbf{u}_m)\ \delta(\mathbf{x}-\mathbf{x}')\,, \tag{20}$$

must be solved. As discussed elsewhere [10], no closed–form solution for $\underline{\underline{M}}(\mathbf{x},\mathbf{x}')$ appears to emerge from (20).

The medium considered here is the most general type of anisotropic dielectric medium for which the determinant operator is the product of two scalar, second–order differential operators of the Helmholtzian type; yet no closed–form expression for $\underline{\underline{G}}(\mathbf{x},\mathbf{x}')$ appears to exist for this medium! Thus, we conclude that

(i) while all currently known closed–form, infinite–medium dyadic Green functions are based on factorizable determinant operators, factorization is not a sufficient condition for the existence of closed–form, infinite–medium, dyadic Green functions; and

(ii) within the class of anisotropic dielectric mediums, the uniaxial dielectric medium (or any medium that can be reduced to such a medium by, for example, affine transformations) remains the most general medium for which a closed–form, infinite–medium, dyadic Green function has been derived.[3]

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[3]Closed–form dyadic Green functions have been found for more general types of anisotropic mediums. Yet, the price paid for a larger parameter space of constitutive parameters is that such formulas only become available when specific algebraic conditions between the parameters are fulfilled. Those conditions appear motivated purely by mathematical necessities and not based on any fundamental symmetry or some other property of the medium.