Onsager–Casimir Principle in the Theory of Bi-Anisotropic Media

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Abstract

In this paper we establish relation between the microscopic and macroscopic Onsager–Casimir principles. It is demonstrated that because of certain symmetries with respect to the space reversal operation, which are intrinsic to any form of the linear constitutive relations, it follows that the symmetries under time reversal expressed by the Onsager–Casimir principle are valid both on the microscopic and macroscopic levels. As one of the possible applications we show that the property of the general bi-anisotropic media to be reciprocal or non-reciprocal does not depend on the constitutive formalism.

1. Introduction

In the electromagnetic theory there is a well-established *Onsager–Casimir principle* or *the principle of the symmetry of kinetic coefficients* [1, 2, 3] which provides certain symmetry relations for microscopic polarizabilities and macroscopic constitutive parameters. The principle is based on the time-reversal symmetry properties of electromagnetic processes. In this paper we show that quite general and important conclusions can be drawn from the fact that microscopic and macroscopic electromagnetic processes possess both time-reversal symmetry (manifested in the Onsager-Casimir principle) and space-reversal symmetry. The latter is manifested by the fact that linear kinetic equations or, in electromagnetics, linear constitutive relations can connect quantities of different mathematical nature. For example, electric polarization is an even vector with respect to the space-reversal operation *(polar vector)*, but magnetic field is odd under this operation *(axial vector)*. Thus, the polarizability coefficient must possess certain mathematical properties, so that operation on an axial vector gives a polar vector. In this paper we consider the two symmetry principles dictated by the time-reversal symmetries and space-reversal symmetries together. In particular, we find that the space-inversion symmetry imposes certain restrictions on the mixing rules which connect microscopic and macroscopic parameters of electromagnetic systems. As one of the applications, we consider the Onsager–Casimir principle for the material parameters of bi-anisotropic media. Here we show that the Onsager–Casimir principle leads always to the same conclusions in all possible formal descriptions of bi-anisotropic media. In particular, if a medium is seen to be reciprocal in one set of material parameters, it is found to be reciprocal in any other set.

2. Theory

2.1 Microscopic and macroscopic Onsager–Casimir principles

On the microscopic level a small bi-anisotropic particle is described by linear relations between the induced electric $\mathbf{p}_e$ and magnetic $\mathbf{p}_m$ dipole moments and external electric $\mathbf{E}$ and magnetic $\mathbf{B}$ fields (in the frequency domain):

$$\mathsf{p} = \mathsf{A} \cdot \mathsf{e} = \begin{pmatrix} \overline{\overline{a}}_{ee} & \overline{\overline{a}}_{em} \\ \overline{\overline{a}}_{me} & \overline{\overline{a}}_{mm} \end{pmatrix} \cdot \mathsf{e}, \quad \text{where} \quad \mathsf{p} = \begin{pmatrix} \mathbf{p}_e \\ \mathbf{p}_m \end{pmatrix}, \quad \mathsf{e} = \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \tag{1}$$

(A small particle is in free space, so $\mathbf{B} = \mu_0 \mathbf{H}$). The Onsager-Casimir relations for the polarizabilities [3, Eqs. (125.14), (125.16)] read:

$$\overline{\overline{a}}_{ee}(\mathbf{H}_0) = \overline{\overline{a}}_{ee}^T(-\mathbf{H}_0), \qquad \overline{\overline{a}}_{mm}(\mathbf{H}_0) = \overline{\overline{a}}_{mm}^T(-\mathbf{H}_0), \qquad \overline{\overline{a}}_{me}(\mathbf{H}_0) = -\overline{\overline{a}}_{em}^T(-\mathbf{H}_0) \tag{2}$$

Here $\mathbf{H}_0$ denotes a medium parameter (or a set of several parameters) which is odd under the time reversal operation.

On the macroscopic level, we deal with the averaged quantities – *constitutive parameters* – which connect the four field quantities as in

$$\mathbf{D} = \overline{\overline{\epsilon}} \cdot \mathbf{E} + \overline{\overline{\xi}} \cdot \mathbf{H}, \qquad \mathbf{B} = \overline{\overline{\mu}} \cdot \mathbf{H} + \overline{\overline{\zeta}} \cdot \mathbf{E} \tag{3}$$

The Onsager-Casimir principle [1, 2, 3] was extended to the constitutive parameters of bi-anisotropic media in [4, 5][1]. Probably the most general proof can be based on the fluctuation-dissipation theorem [3, §124-125]. The proof is straightforward for the constitutive relations (in the frequency domain) written in the following form (for perturbations cased by small fields $\mathbf{D}$ and $\mathbf{B}$):

$$\mathsf{e} = \mathsf{F} \cdot \mathsf{d}, \qquad \text{where} \quad \mathsf{d} = \begin{pmatrix} \mathbf{D} \\ \mathbf{B} \end{pmatrix}, \qquad \mathsf{F} = \begin{pmatrix} \overline{\overline{e}} & \overline{\overline{f}} \\ \overline{\overline{g}} & \overline{\overline{h}} \end{pmatrix} \tag{4}$$

The time-derivative of the energy density is

$$\frac{dW}{dt} = \mathsf{e} \cdot \frac{d\mathsf{d}}{dt} \tag{5}$$

Thus, the same relations follow for the macroscopic parameters as well [4, 5, 8]:

$$\overline{\overline{e}}(\mathbf{H}_0) = \overline{\overline{e}}^T(-\mathbf{H}_0), \qquad \overline{\overline{h}}(\mathbf{H}_0) = \overline{\overline{h}}^T(-\mathbf{H}_0), \qquad \overline{\overline{g}}(\mathbf{H}_0) = -\overline{\overline{f}}^T(-\mathbf{H}_0) \tag{6}$$

For simple dielectrics described in terms of the permittivity tensor only, it is obvious that the same symmetry relation as (6) is also valid for the permittivity [9, §96], because the symmetry properties of the inverse tensor are the same as that of the original tensor.

2.2 Different formalisms

Several forms of material relations can be used to characterize bi-anisotropic media (in the frequency domain). In particular, two such forms are considered in [8]:

$$\mathbf{E} = \overline{\overline{e}} \cdot \mathbf{D} + \overline{\overline{f}} \cdot \mathbf{B}, \qquad \mathbf{H} = \overline{\overline{g}} \cdot \mathbf{D} + \overline{\overline{h}} \cdot \mathbf{B} \tag{7}$$

[1]Probably the first formulation for bi-anisotropic media was published in [6], but an equivalent result in terms of the quantum statistics was published as early as in 1952, see Eqs. (5.3,4) in [7].

(Equations (8) in [8]) and relations (3)

$$\mathbf{D} = \overline{\overline{\epsilon}} \cdot \mathbf{E} + \overline{\overline{\xi}} \cdot \mathbf{H}, \qquad \mathbf{B} = \overline{\overline{\zeta}} \cdot \mathbf{E} + \overline{\overline{\mu}} \cdot \mathbf{H} \tag{8}$$

The Onsager–Casimir principle written in terms of the parameters in (7) reads (6). On the other hand, if relations (8) are used, which means that the macroscopic problem is such that $\mathbf{E}$ and $\mathbf{H}$ can be considered as the primary fields, the same principle gives

$$\overline{\overline{\epsilon}}(\omega, \mathbf{H}_0) = \overline{\overline{\epsilon}}^T(\omega, -\mathbf{H}_0), \qquad \overline{\overline{\mu}}(\omega, \mathbf{H}_0) = \overline{\overline{\mu}}^T(\omega, -\mathbf{H}_0), \qquad \overline{\overline{\xi}}(\omega, \mathbf{H}_0) = -\overline{\overline{\zeta}}^T(\omega, -\mathbf{H}_0) \tag{9}$$

Although the Onsager–Casimir relations look differently for different formalisms, it can be proven that one of them automatically implies the other. Indeed, let us express the material parameters from one set in terms of the other [10]. Consider, for example,

$$\begin{aligned}\overline{\overline{\epsilon}}^T(-\mathbf{H}_0) &= \left[\left(\overline{\overline{e}}(-\mathbf{H}_0) - \overline{\overline{f}}(-\mathbf{H}_0) \cdot \overline{\overline{h}}^{-1}(-\mathbf{H}_0) \cdot \overline{\overline{g}}(-\mathbf{H}_0)\right)^{-1}\right]^T \\ &= \left[\overline{\overline{e}}^T(-\mathbf{H}_0) - \overline{\overline{g}}^T(-\mathbf{H}_0) \cdot (\overline{\overline{h}}^{-1})^T(-\mathbf{H}_0) \cdot \overline{\overline{f}}^T(-\mathbf{H}_0)\right]^{-1}\end{aligned} \tag{10}$$

Using (6) we rewrite this as

$$\overline{\overline{\epsilon}}^T(-\mathbf{H}_0) = \left(\overline{\overline{e}}(\mathbf{H}_0) - \overline{\overline{f}}(\mathbf{H}_0) \cdot \overline{\overline{h}}^{-1}(-\mathbf{H}_0) \cdot \overline{\overline{g}}(\mathbf{H}_0)\right)^{-1} = \overline{\overline{\epsilon}}(\mathbf{H}_0) \tag{11}$$

Similarly, one can prove that the other relations in (6) follow from (9). From this we find that, in contrast to the main conclusion of [8], it is impossible that one of the symmetry relations is satisfied but the other one is not. The same conclusion is true for the Post relations. Effectively, this means that the notion of reciprocity or non-reciprocity always has a clear sense which is independent on the material description formalism.

2.3 Time-reversal and space-reversal symmetries

Artificial bi-anisotropic materials are designed as mixtures of many bi-anisotropic inclusions. Because the composite properties depend on that of the inclusions, we can say that the macroscopic symmetry relations (9) are "inherited" from the microscopic relations (2). Several mixture rules exist which allow to find estimates of the macroscopic parameters from the properties of inclusions. For simple mixture rules, such as the Maxwell Garnett rule, the macroscopic Onsager-Casimir principle can be proved directly from the microscopic relations, because the symmetry properties of the microscopic dyadics dictate the same properties of the macroscopic dyadics. The reason is that different coefficients in (1) and (3) are of different mathematical nature: they behave differently under mirror reflection of spatial coordinates. And this distinction must be preserved in the averaging process which leads to macroscopic constitutive relations.

3. Conclusion

On the microscopic level of consideration we have the fields $\mathbf{E}$ and $\mathbf{B}$ which act on charged particles and cause their displacements. On this level, the fundamental principles such as causality or the symmetry of kinetic coefficients (Onsager–Casimir principle) should be applied to the parameters which connect microscopic responses (such as molecular dipole moments) with the microscopic fields $\mathbf{E}$ and $\mathbf{B}$. Thus, the Fourier transform of the molecular polarizability is subject to the Kramers–Kronig relations, etc. To describe a macroscopic object we average the fields and introduce the notions of so called induction fields **D** and **H**. Instead of the molecular polarizabilities we deal with constitutive parameters which are in a certain sense averaged values of their microscopic counterparts. Naturally, we expect that the same principles of causality and symmetry should also apply to the macroscopic parameters. If not, we would actually conclude that the way of introducing the macroscopic parameters is non-physical. From the above considerations of this paper we in fact can draw an important conclusion that all considered bi-anisotropic material relations are compatible with the Onsager–Casimir principle.

It is important for the macroscopic electromagnetics since at the macroscopic level we cannot any more distinguish between primary and induction fields as the source and reaction fields. In different situations one or the other field can be fixed by the sources and we have to treat that as the primary source field. Causality requirement leads to the conclusion that both the permittivity ϵ and its inverse ϵ^{-1} satisfy the Kramers–Kronig relations. Similarly, both these quantities are even with respect to the time reversal, according to the Onsager–Casimir principle. In the more general case we see that these principles can be universally applied to any of the sets of the bi-anisotropic parameters. If that would not be so, that would actually mean that one or more sets have no physical meaning.

Finally, we have found that the microscopic and macroscopic Onsager–Casimir principles (symmetries with respect to the time inversion) are related, since the macroscopic parameters are introduced as a result of a certain averaging procedure. In this procedure, correct mathematical relations between variables of different mathematical nature should be always maintained.

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