Exact and Approximate Modeling of Helices with Core

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Abstract

A model for the direct calculation of the dipole polarizability tensors of a helix with spherical core based on the exact field solution is introduced. Motivated by this model, an approximate one relying on the dipole moments of the core is developed. Results obtained with both models are compared and verified by measurements.

1. Introduction

Recently, chiral materials consisting of complex inclusions — for example perfectly electrically conducting (PEC) thin wire helices with a dielectric and/or magnetic core different from the host material — were proposed and investigated [1]. In this paper the essentially exact model of a helix with spherical core used for the calculations in [1] is outlined. Because the governing equations are lengthy and their derivation mathematically involved only the Ansatz is displayed here. It reveals a problem concerning cores with both dielectric and magnetic properties. The (approximate) solution of this problem motivates a second model relying on a dipole approximation of the core. This model is developed in the following. Note, that cartesian coordinates are used throughout the whole paper.

2. Model

Consider a helix with spherical core both centered at the coordinate origin. The current flowing on the helix wire can be calculated from a Method of Moments (MoM) solution of the scattering problem [2]. If a thin-wire Galerking MoM is employed, the (symmetric) system matrix $\underline{\underline{Z}}$ and the exciation vector $\underline{V}$ read for plane wave incidence (electric field $\underline{E}^0$)

$$\begin{aligned} Z_{mn} &= j\omega\mu \int_u \int_{u'} \underline{t}(u)(\underline{\underline{G}}^0 + \underline{\underline{G}}^s)\underline{t}'(u')B_m(u')B_n(u)\,du'\,du \\ V_n &= \int_u \underline{t}(u)(\underline{\underline{I}} + \underline{\underline{S}}(u))\underline{E}^0 \exp\{-j\underline{k}\,\underline{r}\}\,B_n(u)\,du \quad . \end{aligned} \tag{1}$$

Here, $\underline{\underline{I}}$ is the unity tensor, $\underline{\underline{G}}^0$ stands for the free space dyadic Greens function, $\underline{\underline{G}}^s$ describes the scattering correction caused by the spherical core [3], $\underline{t}$ ($\underline{t}'$) is the tangential vector on the wire in the observation (source) point, and B stands for the basis/testing functions. The term $\underline{\underline{S}}$ that describes the plane wave scattering of the spherical core can be derived from the scattering correction $\underline{\underline{G}}^s$ by plane wave normalization as described in [4, 5]; it is, in essence, identical to the Mie solution for the scattering of a plane wave by a sphere [6, 7].

The total field scattered by this helix with core can be split into four parts, each corresponding to an individual multipole series. These parts are: 1) The field scattered by the wire caused by the incident plane wave, 2) the field scattered by the core caused by the incident plane wave, 3) the field scattered by the wire caused by the diffraction field of the core, and 4) the field scattered by the core caused by the diffraction field of the wire.

Following the procedure introduced in [8], the incident plane wave in (1) can be expanded into a Taylor series and the different terms arising can be separated into their electric and magnetic origin. This procedure gives results identical to those obtained with the method of counter-propagating waves [9], but removes the information on the direction of incidence analytically instead of using different angles of incidence to separate the polarizabilities. The Taylor series expansion produces unambiguous results if the spherical core is either purely dielectric or purely magnetic. If it possesses both properties, the tight coupling of magnetic and electric fields within the core prohibits an exact solution. One way to circumvent this problem is the following: The inclusion under investigation is intended to form the basic building block of a chiral material; thus, the inclusion itself and all of its constituents must be electrically small. Then, electric and magnetic effects are decoupled nearly entirely (static limit) and can be treated separately by applying the series expansion of the excitation vector $\underline{V}$ (see above).

This idea leads directly to a further simplification of the problem: As the core is electrically small, only the first term of the infinite series defining $\underline{\underline{G}}^s$ contributes significantly [3]. This term represents the dipole contributions to the scattered field. This means that for the description of the interaction between wire and core the latter can effectively be replaced by its electric and magnetic dipole moments in the origin. The expression for the total electric field together with the boundary condition on the PEC wire surface then reads

$$
\begin{aligned}
Z_{mn} &= j\omega \int_u \int_{u'} \left(\mu_0\, \underline{t}\, \underline{\underline{G}}^0\, \underline{t}' - \omega^2\mu_0^2\, \underline{X}\, \underline{\underline{G}}^0(0,u')\, \underline{t}' - \underline{Y}\, \nabla \times \underline{\underline{G}}^0(0,u')\, \underline{t}'\right) B_m(u)\, B_n(u')\, du'\, du \\
V_n &= \int_u \left(\underline{t}\, \underline{E}^0 \exp\{-j\underline{k}\,\underline{r}\} + \omega^2\mu_0\, \underline{X}\, \underline{E}^0(0) - j\omega\, \underline{Y}\, \underline{H}^0(0)\right) B_n(u)\, du \\
\underline{X} &= \underline{t}\, \underline{\underline{G}}^0(u,0)\, \underline{\underline{\alpha}}^e \quad , \quad \underline{Y} = \underline{t}\, \nabla \times \underline{\underline{G}}^0(u,0)\, \underline{\underline{\alpha}}^m \quad ,
\end{aligned}
\tag{2}
$$

where $\underline{\underline{\alpha}}^e$ and $\underline{\underline{\alpha}}^m$ are the symmetric electric and magnetic dipole polarizability tensors of the core, respectively. To ensure reciprocity, the identities $\underline{t}(u) = \underline{t}'(u)$ and $\underline{r}(u) = \underline{r}'(u)$ must be fulfilled when considering V_n as derived in [8].

Together with the abbreviations

$$
\Gamma = \sum_{m,n} B_m(u)\, Z_{mn}^{-1}\, B_n(u') \quad , \quad \langle f(u,u')\rangle = \int_u \int_{u'} f(u,u')\, du'\, du
$$

the expansion of the plane wave contained in (2) now leads to the following dipole polarizabilities of the inclusion (the numbering refers to the different scattered fields discussed above):

$$
\begin{aligned}
\underline{\underline{\alpha}}_1^{ee} &= \frac{1}{j\omega}\langle \underline{t}\,\Gamma\,\underline{t}'\rangle \,, & \underline{\underline{\alpha}}_1^{em} &= -\frac{\mu_0}{2}\langle \underline{t}\,\Gamma\,(\underline{r}'\times\underline{t}')\rangle \\
\underline{\underline{\alpha}}_1^{me} &= \frac{\mu_0}{2}\langle (\underline{r}\times\underline{t})\,\Gamma\,\underline{t}'\rangle \,, & \underline{\underline{\alpha}}_1^{mm} &= -\frac{j\omega\mu_0^2}{4}\langle (\underline{r}\times\underline{t})\,\Gamma\,(\underline{r}'\times\underline{t}')\rangle
\end{aligned}
$$

$$
\underline{\underline{\alpha}}_2^{ee} = \underline{\underline{\alpha}}^e \,, \quad \underline{\underline{\alpha}}_2^{mm} = \underline{\underline{\alpha}}^m
$$

$$
\begin{aligned}
\underline{\underline{\alpha}}_3^{ee} &= -j\omega\mu_0\langle \underline{t}\,\Gamma\,\underline{X}'\rangle \,, & \underline{\underline{\alpha}}_3^{em} &= -\langle \underline{t}\,\Gamma\,\underline{Y}'\rangle \\
\underline{\underline{\alpha}}_3^{me} &= \frac{\omega^2\mu_0^2}{2}\langle (\underline{r}\times\underline{t})\,\Gamma\,\underline{X}'\rangle \,, & \underline{\underline{\alpha}}_3^{mm} &= -\frac{j\omega\mu_0}{2}\langle (\underline{r}\times\underline{t})\,\Gamma\,\underline{Y}'\rangle
\end{aligned}
$$

$$\underline{\underline{\alpha}}_4^{ee} = -j\omega\mu_0\langle \underline{X}\,\Gamma\,\underline{t}'\rangle - j\omega^3\mu_0^2\langle \underline{X}\,\Gamma\,\underline{X}'\rangle\ , \quad \underline{\underline{\alpha}}_4^{em} = -\frac{\omega^2\mu_0^2}{2}\langle \underline{X}\,\Gamma\,(\underline{r}'\times\underline{t}')\rangle - \omega^2\mu_0\langle \underline{X}\,\Gamma\,\underline{Y}'\rangle$$

$$\underline{\underline{\alpha}}_4^{me} = \langle \underline{Y}\,\Gamma\,\underline{t}'\rangle + \omega^2\mu_0\langle \underline{Y}\,\Gamma\,\underline{X}'\rangle\ , \qquad \underline{\underline{\alpha}}_4^{mm} = -\frac{j\omega\mu_0}{2}\langle \underline{Y}\,\Gamma\,(\underline{r}'\times\underline{t}')\rangle - j\omega\langle \underline{Y}\,\Gamma\,\underline{Y}'\rangle$$

Note, that

$$\underline{X} = \underline{t}\,\underline{\underline{G}}^0(u,0)\,\underline{\underline{\alpha}}^e = \underline{\underline{\alpha}}^e\,\underline{\underline{G}}^0(0,u)\,\underline{t}\ ,\quad \underline{Y} = \underline{t}\,\nabla\times\underline{\underline{G}}^0(u,0)\,\underline{\underline{\alpha}}^m = \underline{\underline{\alpha}}^m\,\nabla\times\underline{\underline{G}}^0(0,u)\,\underline{t}\quad .$$

Summing up all these individual contributions, the polarizability tensors of the whole inclusion are obtained. It can be proved, that the resulting tensors represent a reciprocal inclusion, i.e. $\underline{\underline{\alpha}}^{ee} = (\underline{\underline{\alpha}}^{ee})^T$, $\underline{\underline{\alpha}}^{me} = -(\underline{\underline{\alpha}}^{em})^T$, $\underline{\underline{\alpha}}^{mm} = (\underline{\underline{\alpha}}^{mm})^T$.

3. Verification

If the core is purely magnetic or purely dielectric the approximate model can be directly compared to the exact solution. Furthermore, results derived from the system matrix $\underline{\underline{Z}}$ of eq.(1) — for example the resonance frequency of the inclusion — are exact, even for cores with both dielectric and magnetic properties. Thus, these quantities can be compared to the approximate ones obtained from eq.(2) for a partial check of the accuracy. All tests performed showed, that the results calculated from the exact and the approximate model are in very good agreement. As an example, the change in resonance frequency due to a spherical core (diameter: $2a = 3\,\mathrm{mm}$, helix dimensions: diameter 4 mm, height 4 mm, three turns) with magnetic and/or electric properties is displayed in Fig. 1. As can be seen, the errors occurring in the curves for either purely dielectric or purely magnetic cores add up for cores with both properties. As expected, the error increases with electrically larger cores. As a consequence of the neglected electric quadrupole moment this is more pronounced for dielectric cores. Although for ε_r, $\mu_r > 5$ the cores are no longer electrically small ($0.2 \leq k\,a \leq 0.91$), the results are in good agreement, too (maximum difference: 0.04 GHz).

Fig. 2 displays the scattering parameter S_{11} of a helix (dimensions as above) with a spherical dielectric core ($\varepsilon_r = 4.7 - j1.6$, diameter: 3 mm). It was measured using the method described in [10] and calculated using the approximate model. Apart from a small shift in the resonance frequency (see [11] for explanations) the results fit quite well, in particular if one takes into account the measurement accuracy [10].

4. Conclusion

The approximate model for helices with spherical core developed above produces meaningful results, both theoretically (reciprocity) and numerically. Although the model has its origin in the exact solution of the helix/spherical core problem, it contains no restrictions regarding the geometry of the core or the wire. The inclusion should only be electrically small so that higher order multipole moments of the core can be neglected. Thus, the model should be able to handle other than spherical cores, provided their electric and magnetic polarizability tensors are known, either analytically or from numerical calculations.

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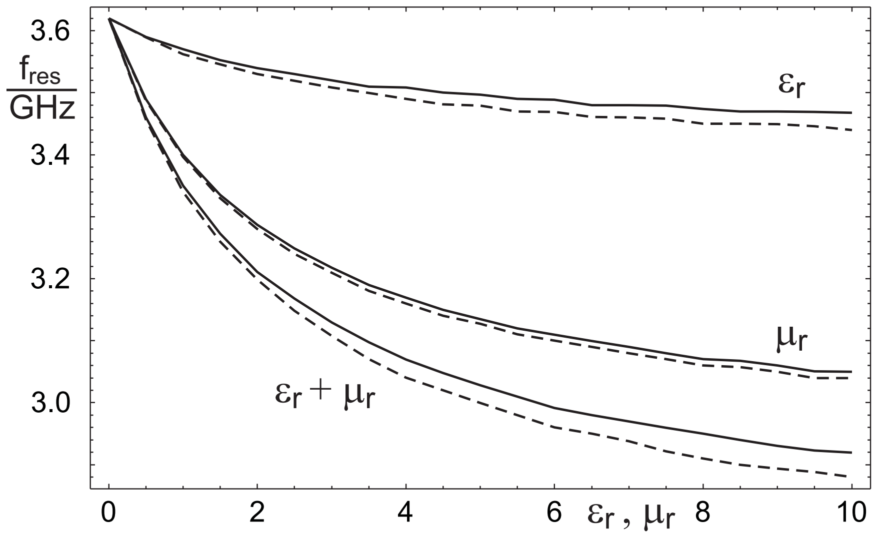

Figure 1: Change in resonance frequency due to a spherical dielectric and/or magnetic core. (——) Approximate model, (– – –) exact model.

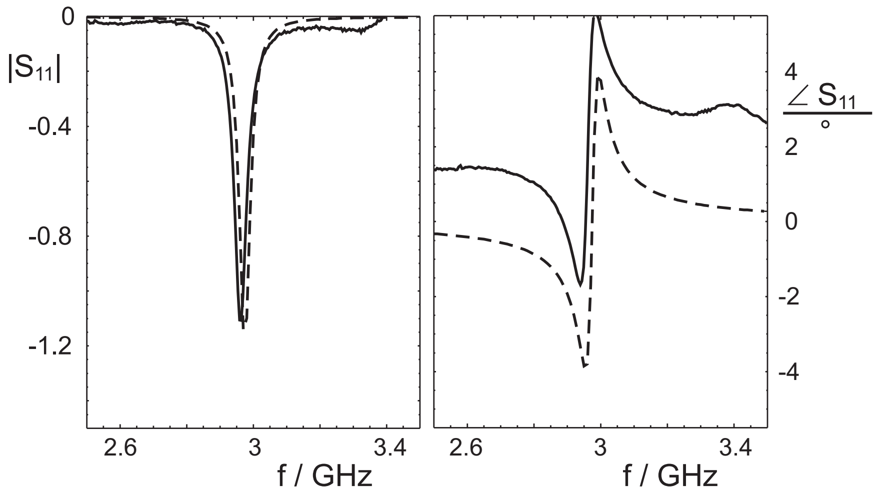

Figure 2: Scattering parameter S_{11} of a helix with dielectric core. Measured (——) and theoretical (– – –) response (approximate model).