A Pedigree of Bianisotropic Media

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Abstract

During the past years we have intensively studied basic properties and field solutions in homogeneous bianisotropic media. We started from simple isotropic media to ever more general bianisotropic media. This study has led to some classification of bianisotropic media, which resembles a pedigree. The pedigree contains two branches. The first branch are so-called self-dual media, these are generalisations of the chiral medium. The second branch are factorizable media, these are generalisations of the uniaxial anisotropic media. We think to have reached some consensus with respect to the pedigree, i.e. we think to have found the most general media in each of the two branches.

1. Introduction

In this contribution we want to report on some of the findings we obtained during the past eight years. During this period we have been investigating homogeneous bianisotropic media with constitutive relations of the form

$$\mathbf{D} = \underline{\epsilon} \cdot \mathbf{E} + \underline{\xi} \cdot \mathbf{H}, \qquad \mathbf{B} = \underline{\zeta} \cdot \mathbf{E} + \underline{\mu} \cdot \mathbf{H}, \tag{1}$$

where $\underline{\epsilon}$, $\underline{\mu}$, $\underline{\zeta}$ and $\underline{\xi}$ are the medium dyadics.

We were interested in finding basic field solutions in these media. The fields for any given source can be found by integration from the fields of an elementary dipole source, i.e. from the Green dyadics. In general it is not possible to obtain the Green dyadics into closed form. Other basic solutions are plane waves. Sometimes an electromagnetic field problem can be simplified by decomposing the fields in two components. Each of these components then propagate in a "simpler" medium for which the Green dyadics are known. Another way to solve field problems is the use of duality transformations which allow us to transform field solutions in one medium to those in another medium. Our aim was to find the most general media for which decomposition of the fields was possible, for which we could find the plane wave solutions and for which the Green dyadics could be obtained in closed form. It turned out that these three problems are intimately interrelated. The common backbone behind these problems is the possibility to factorize the fourth order "Helmholtz determinant operator".

This study resulted in a hierarchical ordening – which we call a pedigree – of ever more general bianisotropic media. Basically this pedigree consists of two separate branches. In the present contribution we will first discuss some basics such as Green dyadics, factorization, plane waves, decomposition and duality. Then we will focus on the the two branches of the pedigree.

This paper gives only a small overview of the subject, a more rigorous historical overview with more references can be found in [1].

2. Green Dyadics and Factorization

The electric Green dyadic $\underline{\underline{G}}_{ee}(\mathbf{r})$ is defined as the relation between the electric current density $\mathbf{J}(\mathbf{r})$ and the electric field $\mathbf{E}(\mathbf{r})$:

$$\mathbf{E}(\mathbf{r}) = -j\omega \iiint\limits_V \underline{\underline{G}}_{ee}(\mathbf{r}-\mathbf{r}') \cdot \mathbf{J}(\mathbf{r}')\mathrm{d}V'. \tag{2}$$

In a bianisotropic medium this Green dyadic satisfies the equation

$$\underline{\underline{H}}_{ee}(\nabla) \cdot \underline{\underline{G}}_{ee}(\mathbf{r}) = -\underline{\underline{I}}\delta(\mathbf{r}), \tag{3}$$

with $\underline{\underline{H}}_{ee}(\nabla)$ the vector Helmholtz operator given by [2]

$$\underline{\underline{H}}_{ee}(\nabla) = -(\nabla \times \underline{\underline{I}} - j\omega\underline{\underline{\xi}}) \cdot \underline{\underline{\mu}}^{-1} \cdot (\nabla \times \underline{\underline{I}} + j\omega\underline{\underline{\zeta}}) + \omega^2\underline{\underline{\epsilon}}. \tag{4}$$

To solve (3) it suffices to find a scalar Green function $G(\mathbf{r})$ that is solution of

$$\mathrm{det}\underline{\underline{H}}_{ee}(\nabla)G(\mathbf{r}) = -\delta(\mathbf{r}), \tag{5}$$

with $\mathrm{det}\underline{\underline{H}}_{ee}(\nabla)$ the Helmholtz determinant operator. The solution of (3) then follows from

$$\underline{\underline{G}}_{ee}(\mathbf{r}) = \left[\underline{\underline{H}}^{A}_{ee}(\nabla)\right]^T G(\mathbf{r}), \tag{6}$$

with $\underline{\underline{H}}^{A}_{ee}(\nabla)$ the adjoint operator of the vector Helmholtz operator. The Helmholtz determinant operator turns out to be a fourth order operator, which makes, in general, a closed form solution of (5) impossible. For some classes of media it is possible to factorize this operator as a product of two second order operators, i.e.

$$\mathrm{det}\underline{\underline{H}}_{ee}(\nabla) = H_a(\nabla)H_b(\nabla). \tag{7}$$

A medium for which this is possible is called factorizable. Factorizability does not necessarily mean that we can solve (5) in closed form. However, often a closed form solution is possible or an elegant series or integral representation.

Sometimes one can factorize the second order dyadic Helmholtz operator:

$$\underline{\underline{H}}_{ee}(\nabla) = \underline{\underline{H}}_a(\nabla) \cdot \underline{\underline{H}}_b(\nabla) \tag{8}$$

where $\underline{\underline{H}}_a(\nabla)$ and $\underline{\underline{H}}_b(\nabla)$ are first order dyadic operators. For these media it is possible to write $\underline{\underline{G}}_{ee}$ in closed form.

3. Plane Waves

For a plane wave of the form $\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 \exp(-j\mathbf{k}\cdot\mathbf{r})$ the vector $\mathbf{E}_0$ satisfies the equation

$$\underline{\underline{H}}_{ee}(-j\mathbf{k}) \cdot \mathbf{E}_0 = 0. \tag{9}$$

A solution different from zero is only possible when $\mathbf{k}$ satisfies the dispersion equation

$$\mathrm{det}\underline{\underline{H}}_{ee}(-j\mathbf{k}) = 0. \tag{10}$$

If we write $\mathbf{k}$ as $k\mathbf{u}$ with $\mathbf{u}$ a unit vector defining the phase velocity propagation direction then (10) is a fourth order polynomial equation in k and the dispersion surface will be a fourth order surface. When $\mathrm{det}\underline{\underline{H}}_{ee}$ is factorizable the dispersion surface consists of two second order surfaces, i.e. of two quadrics. For a given value of $\mathbf{k}$ the solution of equation (9) gives the polarization of the plane waves.

4. Decomposition

With decomposition we mean that the fields can be split in two components as

$$\mathbf{E} = \mathbf{E}_a + \mathbf{E}_b, \qquad \mathbf{H} = \mathbf{H}_a + \mathbf{H}_b. \tag{11}$$

Both a and b are solution of Maxwell equations for simpler media than the original medium. These simpler media are called the equivalent media. The most well known field decomposition is the TE and TM decomposition for uniaxial anisotropic media [3] for which the equivalent media are isotropic media. This can be generalised to media where the decomposed fields satisfy the conditions

$$\mathbf{a}_1 \cdot \mathbf{E}_a + \mathbf{a}_2 \cdot \mathbf{H}_a = 0, \qquad \mathbf{b}_1 \cdot \mathbf{E}_b + \mathbf{b}_2 \cdot \mathbf{H}_b = 0, \tag{12}$$

where $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{b}_1$ and $\mathbf{b}_2$ are arbitrary vectors.

Another way to decompose the fields is the Bohren decomposition [4]. In this case the fields are decomposed as

$$\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_-, \qquad \mathbf{H} = \mathbf{H}_+ + \mathbf{H}_-, \tag{13}$$

with

$$\mathbf{E}_\pm = Y_\pm \mathbf{H}_\pm, \tag{14}$$

with $Y_\pm$ some scalar constants. The most well known medium that allows such a decomposition is the isotropic chiral medium. Also in this case there are much more general media that allow a Bohren decomposition.

5. Duality

A duality transformation transforms original fields $\mathbf{E}$ and $\mathbf{H}$ into dual fields $\mathbf{E}_d$ and $\mathbf{H}_d$ as

$$\begin{pmatrix} \mathbf{E}_d \\ \mathbf{H}_d \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}, \tag{15}$$

with A, B, C and D arbitrary constants. The dual fields satisfy Maxwell equations in a dual medium defined by its medium dyadics as

$$\begin{pmatrix} \underline{\epsilon}_d & \underline{\xi}_d \\ \underline{\zeta}_d & \underline{\mu}_d \end{pmatrix} = - \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \underline{\epsilon} & \underline{\xi} \\ \underline{\zeta} & \underline{\mu} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1}. \tag{16}$$

Using the duality transformation it is possible to transform field solutions from the original medium to the dual medium or vice versa. For example the Green dyadics between both media are related through [5]

$$\begin{pmatrix} \underline{G}_{ee,d} & \underline{G}_{em,d} \\ \underline{G}_{me,d} & \underline{G}_{mm,d} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \underline{G}_{ee} & \underline{G}_{em} \\ \underline{G}_{me} & \underline{G}_{mm} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1}. \tag{17}$$

When a medium is invariant under a duality transformation we say it is a self-dual medium. Only a restricted class of bianisotropic media are self-dual.

6. The First Branch

It turns out that the class of media that are self-dual or that allow a Bohren decomposition or that allow a factorization of the vector Helmholtz operator is one and the same. These media have been studied in [6]–[11]. The isotropic chiral medium is the simpliest representative of this class. The most general medium of this class is given by

$$\underline{\epsilon} = \epsilon\underline{\alpha}, \qquad \underline{\mu} = \mu\underline{\alpha}, \qquad \underline{\zeta} = (\chi\underline{\alpha} + j\underline{\kappa})\sqrt{\epsilon\mu}, \qquad \underline{\xi} = (\chi\underline{\alpha} - j\underline{\kappa})\sqrt{\epsilon\mu}, \tag{18}$$

with $\underline{\alpha}$ and $\underline{\kappa}$ arbitrary dyadics.

The Green dyadics for this medium can be written in closed form [11] as follows

$$\begin{aligned}\underline{G}_{ee}(\mathbf{r}) = &-\frac{j\eta\exp(k\mathbf{a}_+\cdot\mathbf{r})}{2\cos\theta}\underline{L}_+(\nabla)\left(\frac{\exp(-jkD_+)}{4\pi kD_+}\right)\\ &-\frac{j\eta\exp(k\mathbf{a}_-\cdot\mathbf{r})}{2\cos\theta}\underline{L}_-(\nabla)\left(\frac{\exp(-jkD_-)}{4\pi kD_-}\right),\end{aligned} \tag{19}$$

with

$$k = \omega\sqrt{\epsilon\mu}, \quad \eta = \sqrt{\frac{\mu}{\epsilon}}, \quad D_\pm = \sqrt{\det\underline{S}_\pm}\sqrt{\mathbf{r}\cdot\underline{S}_\pm^{-1}\cdot\mathbf{r}}, \quad \cos\theta = \sqrt{1-\chi^2}, \tag{20}$$

$$\underline{L}_\pm(\nabla) = \nabla\nabla \pm k(\nabla\cdot\underline{S}_\pm)\times\underline{I} + k^2\underline{S}_\pm^{-1}\det\underline{S}_\pm \tag{21}$$

and where $\underline{S}_\pm$ and $\mathbf{a}_\pm$ follow from a decomposition of $\cos\theta\underline{\alpha} \pm \underline{\kappa}$ in a symmetric and an asymmetric part of the form

$$\cos\theta\underline{\alpha} \pm \underline{\kappa} = \underline{S}_\pm + \mathbf{a}_\pm \times \underline{I}. \tag{22}$$

For more information on the media in this branch we also refer to another paper in these proceedings [12].

7. The Second Branch

The second branch is much more complicated. This branch contains the media that are decomposable. It turns out that these media are also factorizable. It took us a long while to find the medium that has all decomposable and factorisable media as special cases. This medium is described by rather complicated medium parameters [13]

$$\underline{\epsilon} = \alpha(\mathbf{z}\times\underline{I} + \mathbf{a}_1\mathbf{b}_1 + \mathbf{b}_1\mathbf{a}_1) + \eta(-\underline{B}^T + \mathbf{a}_2\mathbf{b}_1 + \mathbf{b}_2\mathbf{a}_1), \tag{23}$$

$$\underline{\xi} = \eta(\mathbf{x}\times\underline{I} + \mathbf{a}_2\mathbf{b}_2 + \mathbf{b}_2\mathbf{a}_2) + \alpha(\underline{B} + \mathbf{a}_1\mathbf{b}_2 + \mathbf{b}_1\mathbf{a}_2), \tag{24}$$

$$\underline{\zeta} = \tau(\mathbf{z}\times\underline{I} + \mathbf{a}_1\mathbf{b}_1 + \mathbf{b}_1\mathbf{a}_1) - \alpha(-\underline{B}^T + \mathbf{a}_2\mathbf{b}_1 + \mathbf{b}_2\mathbf{a}_1), \tag{25}$$

$$\underline{\mu} = -\alpha(\mathbf{x}\times\underline{I} + \mathbf{a}_2\mathbf{b}_2 + \mathbf{b}_2\mathbf{a}_2) + \tau(\underline{B} + \mathbf{a}_1\mathbf{b}_2 + \mathbf{b}_1\mathbf{a}_2), \tag{26}$$

where α, η, τ, $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{x}$, $\mathbf{z}$ and $\underline{B}$ are arbitrary scalars, vectors and dyadics. It turns out that this medium is also closed with respect to duality transformations. This means that it is not possible to generalize this medium further using a duality transformation. These observations made us conclude that (23)–(26) is the most general medium that allows decomposition and factorisation.

Special cases of the medium (23)–(26) were studied in [14]–[34]. An important special case are the anisotropic media. The most general anisotropic medium that allows factorization and decomposition is $\underline{\epsilon} = \tau\underline{\mu}^T + \mathbf{ab}$ [24], with as special case the uniaxial anisotropic medium.

Another interesting class of media are the equivalent media of the medium (23)–(26). These are of the form [13]

$$\underline{\epsilon} = -\eta\underline{B}^T + \alpha(\mathbf{z}\times\underline{I}), \qquad \underline{\mu} = \tau\underline{B} - \alpha(\mathbf{x}\times\underline{I}), \tag{27}$$

$$\underline{\xi} = \alpha\underline{B} + \eta(\mathbf{x}\times\underline{I}), \qquad \underline{\zeta} = \alpha\underline{B}^T + \tau(\mathbf{z}\times\underline{I}). \tag{28}$$

Without loss of generality we assume that $\tau\eta + \alpha^2 = 1$. The Helmholtz determinant operator for these media can be written as a square. This allows a closed form Green dyadic given by

$$\underline{G}_{ee}(\mathbf{r}) = \{\frac{\alpha}{j\omega}[(\alpha\nabla \times \underline{I} - j\omega\underline{B}) \cdot (\tau\nabla - j\omega\mathbf{x})] \times \underline{I}$$

$$-\frac{\tau}{j\omega}[(\eta\nabla + j\omega\mathbf{z})(\tau\nabla - j\omega\mathbf{x}) + \alpha^2\nabla\nabla + j\omega\alpha\nabla(\underline{B}^{T} \overset{\times}{\cdot} \underline{I}) + \underline{I} \times (\nabla \cdot \underline{B}) - \omega^2(\underline{B}^{-1})^T \mathrm{det}\underline{B}]\}$$

$$\frac{e^{j\omega\mathbf{c}\cdot\underline{R}^{-1}\cdot\mathbf{r}} e^{-jkD}}{j\omega 4\pi\sqrt{\mathrm{det}\underline{R}}\sqrt{D}}, \tag{29}$$

with

$$\underline{R} = \frac{1}{2}(\alpha^2 + \tau\eta)(\underline{B} + \underline{B}^T), \quad \mathbf{c} = \frac{1}{2}[\eta\underline{B} \cdot \mathbf{x} + \tau\mathbf{z} \cdot \underline{B} - \alpha\mathbf{z} \times \mathbf{x} + \alpha(\mathrm{det}\underline{B})(\underline{B}^{-1})^{T} \overset{\times}{\cdot} \underline{I}], \tag{30}$$

and

$$k = \omega\sqrt{\mathbf{z} \cdot \underline{B} \cdot \mathbf{x} + \mathrm{det}\underline{B} - \mathbf{c} \cdot \underline{R}^{-1} \cdot \mathbf{c}}, \quad D = \sqrt{\mathbf{r} \cdot \underline{R}^{-1} \cdot \mathbf{r}}, \tag{31}$$

where $\underline{A} \overset{\times}{\cdot} \underline{I}$ is shorthand for $\mathbf{u}_x \times \underline{A} \cdot \mathbf{u}_x + \mathbf{u}_y \times \underline{A} \cdot \mathbf{u}_y + \mathbf{u}_z \times \underline{A} \cdot \mathbf{u}_z$. The fact that the Helmholtz determinant operator is a square also means that the two second order dispersion surfaces coincide. Each of the equivalent media of a certain original medium have as coinciding dispersion surfaces one of the two dispersion surfaces of the original medium.

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