Numerical Homogenization Studies of Biaxial Bianisotropic Composite Materials

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Abstract

We explore the conceptualization of biaxial composite mediums through the process of homogenization. Biaxiality is found to arise when the component mediums undergoing homogenization present two non–collinear distinguished axes. Two possible sources of directionality in the component mediums are considered: (a) topological and (b) electromagnetic. Examples of these are investigated by considering the homogenization of particulate components with (a) non–spherical topologies and isotropic electromagnetic properties and (b) uniaxial electromagnetic properties and spherical topologies.

1. Introduction

In the context of electromagnetic material properties, the concept of *homogenization* is both scientifically and technologically important. Composite mediums with complex properties may be conceptualized through homogenizing relatively simple constituent mediums. Biaxial symmetry in such homogenized composite mediums (HCMs) is our primary concern here. We build upon the foundation laid by earlier studies of non–dissipative dielectric [1] and dissipative dielectric–magnetic [2] biaxial HCMs (wherein further background details may be found) and generalize to the bianisotropic case. By considering only component mediums of the simplest forms, we demonstrate that an elaborate HCM form can arise; and through illustrative parametric studies, we delineate symmetries in the HCM structure.

2. Preliminaries

We consider HCMs derived from only two (distinct) component mediums, each being envisioned in particulate form; we refer to them as the *host* medium and *inclusion* medium. Of the many formalisms which have been developed in order to estimate the electromagnetic constitutive properties of HCMs, here we adopt the Bruggeman formalism [3, 4, 5]. The HCMs emerging from the numerical calculations may be characterized by the bianisotropic constitutive relations[1]

$$\mathbf{D}(\mathbf{x}) = \epsilon_0 \underline{\underline{\epsilon}}_{HCM} \bullet \mathbf{E}(\mathbf{x}) + \sqrt{\epsilon_0\mu_0}\, \underline{\underline{\xi}}_{HCM} \bullet \mathbf{H}(\mathbf{x}), \tag{1}$$

$$\mathbf{B}(\mathbf{x}) = \sqrt{\epsilon_0\mu_0}\, \underline{\underline{\zeta}}_{HCM} \bullet \mathbf{E}(\mathbf{x}) + \mu_0 \underline{\underline{\mu}}_{HCM} \bullet \mathbf{H}(\mathbf{x}). \tag{2}$$

[1]Vector quantities are in boldface while dyadics are double underlined. The unit dyadic is denoted by $\underline{\underline{I}}$ and $(\mathbf{u_x}, \mathbf{u_y}, \mathbf{u_z})$ is the triad of cartesian unit vectors. The permittivity and permeability of free space are denoted by ϵ_0 and μ_0, respectively.

The HCM constitutive dyadics have a biaxial form which we represent as [1, 2, 6]

$$\underline{\underline{\tau}}_{HCM} = a_\tau^r \underline{\underline{I}} + b_\tau^r (\mathbf{u}_{m\tau}^r \mathbf{u}_{n\tau}^r + \mathbf{u}_{n\tau}^r \mathbf{u}_{m\tau}^r) + i \left[a_\tau^i \underline{\underline{I}} + b_\tau^i (\mathbf{u}_{m\tau}^i \mathbf{u}_{n\tau}^i + \mathbf{u}_{n\tau}^i \mathbf{u}_{m\tau}^i) \right], \quad (\tau = \epsilon, \xi, \zeta, \mu), \tag{3}$$

where $a_\tau^{r,i}$ and $b_\tau^{r,i}$ are real–valued scalars and we describe the real–valued unit vectors in terms of spherical polar coordinates as

$$\mathbf{u}_{\kappa\tau}^\chi = \sin\theta_{\kappa\tau}^\chi \cos\phi_{\kappa\tau}^\chi \, \mathbf{u}_x + \sin\theta_{\kappa\tau}^\chi \sin\phi_{\kappa\tau}^\chi \, \mathbf{u}_y + \cos\theta_{\kappa\tau}^\chi \, \mathbf{u}_z, \quad (\chi = r, i; \;\; \kappa = m, n; \;\; \tau = \epsilon, \xi, \zeta, \mu). \tag{4}$$

With one exception, we consider here only constituent mediums with distinguished axes lying in the xy plane. Consequently, our calculations reveal that the HCM unit vector pairs $\mathbf{u}_{mc}^\chi$ and $\mathbf{u}_{nc}^\chi$ always lie in planes perpendicular to the xy plane with the xy plane bisecting the angle between $\mathbf{u}_{mc}^\chi$ and $\mathbf{u}_{nc}^\chi$. The following identities therefore hold

$$\theta_{m\tau}^\chi = \pi - \theta_{n\tau}^\chi = \theta_\tau^\chi, \qquad \phi_{m\tau}^\chi = \phi_{n\tau}^\chi = \phi_\tau^\chi, \qquad (\chi = r, i; \;\; \tau = \epsilon, \xi, \zeta, \mu). \tag{5}$$

The one exception occurs when we consider ellipsoidal inclusions of varying eccentricity; in this case one distinguished axis can lie along the z axis and we shall treat this as a special case in Section 3.1. Furthermore, since all component mediums we consider are reciprocal, results for the magnetoelectric dyadic $\underline{\underline{\zeta}}_{HCM}$ need not be explicitly presented as we find $\underline{\underline{\xi}}_{HCM} = -\underline{\underline{\zeta}}_{HCM}$.

All graphs are presented with reference to the key given in Table 1. A volumetric proportion of inclusion medium to host medium of 0.3 is taken for all calculations.

3. Numerical Homogenization Calculations

3.1 Dielectric case

We homogenize a host medium of permittivity $\underline{\underline{\epsilon}}^{host} = 1.2\,\underline{\underline{I}}$ and spherical topology with an inclusion medium of permittivity $\underline{\underline{\epsilon}}^{inc} = (3 + 3i)\,\underline{\underline{I}}$ and ellipsoidal geometry characterized by the shape dyadic $\underline{\underline{U}}^{inc} = \mathrm{diag}(2, 1, \gamma)$. For this particular example (and no others) the identities (5) do not hold; instead we take $\theta_{m\epsilon}^{r,i} = \theta_\epsilon^{r,i}$ and $\phi_{m\epsilon}^{r,i} = \phi_\epsilon^{r,i}$ and plot these angles as a function of γ in Figure 1. At points where the inclusion shape becomes spheroidal, the unit vectors $\mathbf{u}_{m\epsilon,n\epsilon}^{r,i}$ all lie on a common axis and the HCM becomes uniaxial. For all values of γ we find that $\underline{\underline{\epsilon}}_{HCM}$ is diagonal and hence the HCM belongs to the biaxial orthorhombic class [7]. This reflects the fact that in this case biaxiality arises from a geometrical structure based on three mutually perpendicular principal axes, namely those of the inclusion ellipsoid.

3.2 Dielectric–magnetic case

Here we consider constituents in which the distinguished axes have an electromagnetic, rather than topological, origin. We homogenize a host medium with constitutive dyadics $\underline{\underline{\epsilon}}^{host} = \underline{\underline{\mu}}^{host} = \mathrm{diag}(3, 1, 1)$ and an inclusion medium specified by $\underline{\underline{\epsilon}}^{inc} = (1 + i)\,\underline{\underline{A}}$ and $\underline{\underline{\mu}}^{inc} = (2 + i)\,\underline{\underline{A}}$ where

$$\underline{\underline{A}} = \begin{bmatrix} 3\cos^2\lambda + \sin^2\lambda & 2\sin\lambda\cos\lambda & 0 \\ 2\sin\lambda\cos\lambda & 3\sin^2\lambda + \cos^2\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{6}$$

and both host and inclusion mediums have a spherical topology. Plotted in Figures 2 and 3 as functions of λ are, respectively, the defining angles for the unit vector pairs $\mathbf{u}_{m\epsilon,n\epsilon,m\mu,n\mu}^{r,i}$ and the corresponding permittivity ($a_\epsilon^{r,i}$ and $b_\epsilon^{r,i}$) and permeability scalars ($a_\mu^{r,i}$ and $b_\mu^{r,i}$) . A uniaxial dielectric–magnetic HCM results when the distinguished axes of the constituent mediums are aligned. With the exception of the special cases $\lambda = 0,\, \pi/2$ and π, all eight angles $\theta_{\epsilon,\mu}^{r,i}$ and $\phi_{\epsilon,\mu}^{r,i}$ and all eight scalars $a_{\epsilon,\mu}^{r,i}$ and $b_{\epsilon,\mu}^{r,i}$ have distinct values and the biaxial HCM is of the monoclinic/triclinic type as regards both $\underline{\underline{\epsilon}}_{HCM}$ and $\underline{\underline{\mu}}_{HCM}$ [7].

We repeat the homogenizations of Figures 2 and 3 but now with a more general host medium characterized by $\underline{\underline{\epsilon}}^{host} = \underline{\underline{\mu}}^{host} = \mathrm{diag}(3+3i\delta, 1+i\delta, 1+i\delta)$ and with a fixed angle for the inclusion distinguished axis of $\lambda = 50°$. The angles $\theta_{\epsilon,\mu}^{r,i}$ are plotted against δ in Figure 4. Our findings for this example may be summarized by:

$$\mathrm{Re}\,\underline{\underline{\tau}}^{host} = p_\tau\,\mathrm{Im}\,\underline{\underline{\tau}}^{host}, \quad \mathrm{Re}\,\underline{\underline{\tau}}^{inc} = p_\tau\,\mathrm{Im}\,\underline{\underline{\tau}}^{inc} \quad \Rightarrow \quad \theta_\tau^r = \theta_\tau^i, \quad \phi_\tau^r = \phi_\tau^i, \quad (\tau = \epsilon,\, \mu), \tag{7}$$

where $p_{\epsilon,\,\mu}$ are proportionality scalars. Thus, the biaxial HCM structure becomes orthorhombic with respect to permittivity (permeability) when ratios of real and imaginary parts of $\underline{\underline{\epsilon}}^{host}$ and $\underline{\underline{\epsilon}}^{inc}$ ($\underline{\underline{\mu}}^{host}$ and $\underline{\underline{\mu}}^{inc}$) are equal, despite the distinguished axes of the constituent mediums being non–perpendicular.

3.3 Bianisotropic case

Finally we consider the general bianisotropic case: the homogenization of an inclusion medium characterized by $\underline{\underline{\epsilon}}^{inc} = 2(1+i)\,\underline{\underline{A}}$, $\underline{\underline{\mu}}^{inc} = 1.5(1+i)\,\underline{\underline{A}}$ and $\underline{\underline{\xi}}^{inc} = -\underline{\underline{\zeta}}^{inc} = (1+i)\underline{\underline{A}}$ where $\lambda = 50°$, with a host medium described by $\underline{\underline{\epsilon}}^{host} = \underline{\underline{\mu}}^{host} = \underline{\underline{\xi}}^{host} = -\underline{\underline{\zeta}}^{host} = diag(3+3i\delta, 1+i\delta, 1+i\delta)$; and spherical topology is chosen for both component mediums. The computed biaxial bianisotropic HCM structure is of the generalized monoclinic/triclinic type with angles $\theta_{\epsilon,\xi,\mu}^{r,i}$ and $\phi_{\epsilon,\xi,\mu}^{r,i}$, and scalars $a_{\epsilon,\xi,\mu}^{r,i}$ and $b_{\epsilon,\xi,\mu}^{r,i}$, all taking distinct values, in general. The corresponding polar HCM unit vector angles are displayed as functions of δ in Figure 5 (the azimuthal angles behave similarly). At the point $\delta = 1$ we find that the HCM is orthorhombic biaxial with respect to all four constitutive dyadics and, as in Section 3.2, the orthorhombic state is not associated with perpendicularity of the distinguished axes in the component mediums.

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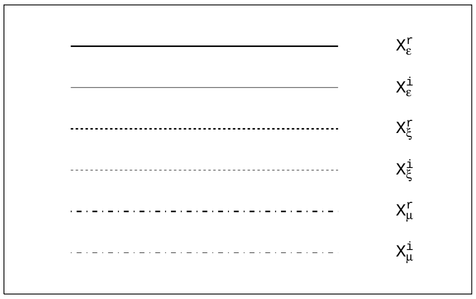

Table1: Key for Figures 1–5. $X = a, b, \theta$ or ϕ.

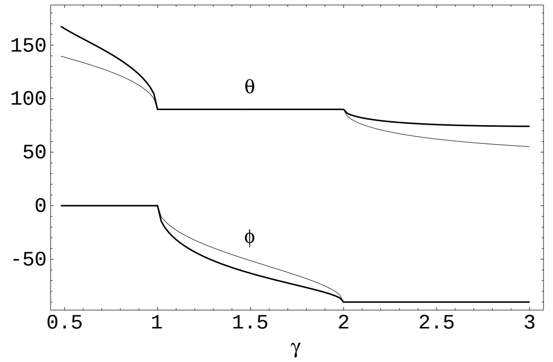

Figure 1: HCM angles $\theta_\epsilon^{r,i}$ and $\phi_\epsilon^{r,i}$ vs. inclusion ellipsoid semi–axis γ. ($X = \theta$, ϕ in Table 1).

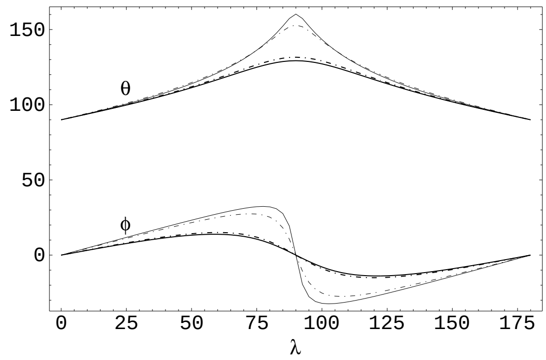

Figure 2: HCM angles $\theta_{\epsilon,\mu}^{r,i}$ and $\phi_{\epsilon,\mu}^{r,i}$ vs. inclusion distinguished axis angle λ. ($X = \theta$, ϕ in Table 1).

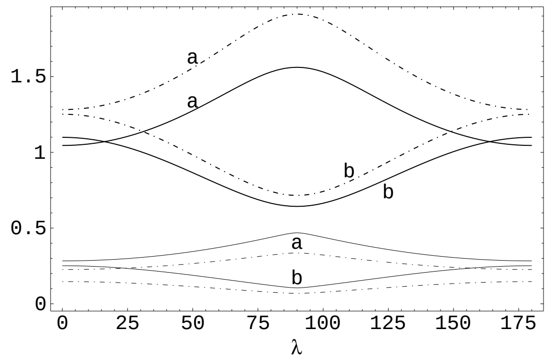

Figure 3: HCM scalars $a_{\epsilon,\mu}^{r,i}$ and $b_{\epsilon,\mu}^{r,i}$ vs. inclusion distinguished axis angle λ. ($X = a$, b in Table 1).

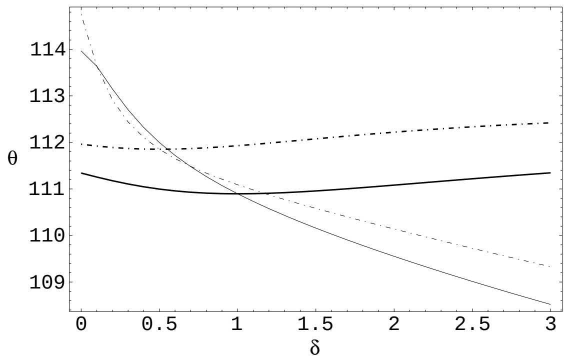

Figure 4: HCM angles $\theta_{\epsilon,\mu}^{r,i}$ vs. host medium parameter δ. ($X = \theta$ in Table 1).

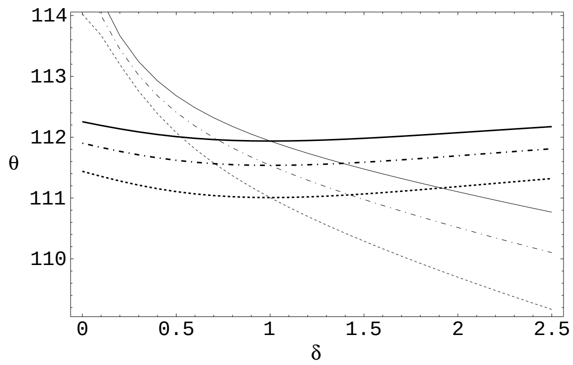

Figure 5: HCM angles $\theta_{\epsilon,\xi,\mu}^{r,i}$ vs. host medium parameter δ. ($X = \theta$ in Table 1).