Green Dyadics for Self–Dual Bi–Anisotropic Media

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Abstract

The class of self-dual linear bi-anisotropic media can be defined in three different ways. It consists of media which are invariant in a duality transformation, allow factorization of the second-order dyadic Helmholtz operator in terms of two first-order dyadic operators and allow decomposition of fields and sources in a way that is an extension of the Bohren decomposition for chiral media. It is shown that the Green dyadic can be solved in closed analytic form for any self-dual bi-anisotropic medium and its general expression is given in terms of the self-dual decomposition.

1. Introduction

The Green dyadic corresponding to the homogeneous space of a bi-anisotropic medium gives the field from an arbitrary point source in that medium and forms the basis for all computations of electromagnetic field problems. It characterizes the dependence of the electromagnetic field on the medium in question. Presently, an analytic expression of the Green dyadic is known for a very limited number of linear bi-anisotropic media. It is of interest to increase the number of media with analytic Green dyadic solutions.

The media under consideration in this study are defined by the four medium dyadics appearing in the constitutive equations

$$\begin{pmatrix} \mathbf{D} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} \overline{\overline{\epsilon}} & \overline{\overline{\xi}} \\ \overline{\overline{\zeta}} & \overline{\overline{\mu}} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}, \tag{1}$$

restricted by the general form

$$\begin{pmatrix} \overline{\overline{\epsilon}} & \overline{\overline{\xi}} \\ \overline{\overline{\zeta}} & \overline{\overline{\mu}} \end{pmatrix} = \begin{pmatrix} \epsilon & \sqrt{\mu\epsilon}\sin\theta \\ \sqrt{\mu\epsilon}\sin\theta & \mu \end{pmatrix} \overline{\overline{\alpha}} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} j\sqrt{\mu\epsilon}\,\overline{\overline{\kappa}}_r. \tag{2}$$

Here, the dimensionless dyadics $\overline{\overline{\alpha}}$ and $\overline{\overline{\kappa}}_r$ are arbitrary. For convenience we also denote $\eta = \sqrt{\mu/\epsilon}$ and $k = \omega\sqrt{\mu\epsilon}$. The class of bi-anisotropic media (2) under consideration appears quite interesting because it can be derived in three different ways considered below.

2. Self–Dual Media

Defining the duality transformation for the electromagnetic fields as the linear mapping

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \Rightarrow \begin{pmatrix} \mathbf{E}_d \\ \mathbf{H}_d \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}, \quad AD - BC \neq 0, \tag{3}$$

for the medium dyadics it induces the transformation [1]

$$\begin{pmatrix} \overline{\overline{\epsilon}}_d \\ \overline{\overline{\xi}}_d \\ \zeta_d \\ \overline{\overline{\mu}}_d \end{pmatrix} = \frac{1}{AD-BC} \begin{pmatrix} D^2 & -CD & -CD & C^2 \\ -BD & AD & BC & -AC \\ -BD & BC & AD & -AC \\ B^2 & -AB & -AB & A^2 \end{pmatrix} \begin{pmatrix} \overline{\overline{\epsilon}} \\ \overline{\overline{\xi}} \\ \overline{\overline{\zeta}} \\ \overline{\overline{\mu}} \end{pmatrix}. \tag{4}$$

It can now be shown that (2) describes the class of media invariant in the duality transformation, by assuming first that the four medium dyadics satisfy the self-dual conditions $\overline{\overline{\epsilon}}_d = \overline{\overline{\epsilon}}$, $\overline{\overline{\xi}}_d = \overline{\overline{\xi}}$, $\overline{\overline{\zeta}}_d = \overline{\overline{\zeta}}$ and $\overline{\overline{\mu}}_d = \overline{\overline{\mu}}$ in a particular duality transformation defined by four parameters $A - D$. This leads to two independent equations

$$(A-D)\overline{\overline{\epsilon}} + C(\overline{\overline{\xi}} + \overline{\overline{\zeta}}) = 0, \qquad (D-A)\overline{\overline{\mu}} + B(\overline{\overline{\xi}} + \overline{\overline{\zeta}}) = 0 \tag{5}$$

implying that the dyadics $\overline{\overline{\epsilon}}$, $\overline{\overline{\mu}}$ and $\overline{\overline{\xi}} + \overline{\overline{\zeta}}$ must be multiples of the same dyadic, say, $\overline{\overline{\alpha}}$. There is no condition to the dyadic $\overline{\overline{\xi}} - \overline{\overline{\zeta}}$, which can be chosen independently as a multiple of another dyadic $\overline{\overline{\kappa}}_r$. Thus, the medium dyadics of a bi-anisotropic medium self dual in a specific duality transformation must be of the form (2) which is why the class is labeled as that of self-dual media.

3. Factorizable Helmholtz Operator

A second way to define the class of media (2) is by requiring that the second-order dyadic Helmholtz operator [2]

$$\overline{\overline{H}}_e(\nabla) = -(\nabla \times \overline{\overline{I}} - j\omega\overline{\overline{\xi}}) \cdot \overline{\overline{\mu}}^{-1} \cdot (\nabla \times \overline{\overline{I}} + j\omega\overline{\overline{\zeta}}) + \omega^2\overline{\overline{\epsilon}} \tag{6}$$

be factorizable in terms of two first-order dyadic operators. For two operators to be the same, they have to coincide termwise in all orders of differentiation. Let us look for a factorization of the form

$$\overline{\overline{H}}_e(\nabla) = -(\nabla \times \overline{\overline{I}} + \overline{\overline{A}}) \cdot \overline{\overline{\mu}}^{-1} \cdot (\nabla \times \overline{\overline{I}} + \overline{\overline{B}}), \tag{7}$$

because in this case the second-order terms of (6) and (7) are identically the same. By equating the first-order terms of (6) and (7) one finds the dyadics must be of the form $\overline{\overline{B}} = \alpha\overline{\overline{\mu}} + j\omega\overline{\overline{\zeta}}$, $\overline{\overline{A}} = \beta\overline{\overline{\mu}} - j\omega\overline{\overline{\xi}}$ with $\beta = -\alpha$. Equating the zeroth-order terms of (6) and (7) leads to a dyadic equation for the scalar α:

$$\alpha^2\overline{\overline{\mu}} + j\omega\alpha(\overline{\overline{\xi}} + \overline{\overline{\zeta}}) - \omega^2\overline{\overline{\epsilon}} = 0. \tag{8}$$

Now it is obvious that this has no solution unless the three dyadics $\overline{\overline{\mu}}$, $\overline{\overline{\epsilon}}$ and $\overline{\overline{\xi}} + \overline{\overline{\zeta}}$ are multiples of the same dyadic and, again, $\overline{\overline{\xi}} - \overline{\overline{\zeta}}$ can be any dyadic. Thus, we have rearrived at the class of self-dual media. Assuming medium dyadic expressions of the form (2), the solution can be written as $\alpha = \omega\sqrt{\epsilon/\mu}\; e^{-j\theta}$ and the electric Helmholtz operator dyadic has the simple factorization

$$\overline{\overline{H}}_e(\nabla) = -\overline{\overline{H}}_+(\nabla) \cdot \overline{\overline{\mu}}^{-1} \cdot \overline{\overline{H}}_-(\nabla) = -\overline{\overline{H}}_-(\nabla) \cdot \overline{\overline{\mu}}^{-1} \cdot \overline{\overline{H}}_+(\nabla), \tag{9}$$

in terms of two first-order operators defined by

$$\overline{\overline{H}}_\pm(\nabla) = \nabla \times \overline{\overline{I}} \mp k\overline{\overline{\tau}}_\pm, \qquad \overline{\overline{\tau}}_\pm = \cos\theta\, \overline{\overline{\alpha}} \pm \overline{\overline{\kappa}}_r. \tag{10}$$

4. Decomposable Fields

Finally, the same class of self-dual media can be defined by requiring that the fields be decomposable in two independent electromagnetic fields as

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_+(\mathbf{r}) + \mathbf{E}_-(\mathbf{r}), \qquad \mathbf{H}(\mathbf{r}) = \mathbf{H}_+(\mathbf{r}) + \mathbf{H}_-(\mathbf{r}), \tag{11}$$

with scalar admittance relations necessary for the decomposition

$$\mathbf{H}_+(\mathbf{r}) = Y_+\mathbf{E}_+(\mathbf{r}), \qquad \mathbf{H}_-(\mathbf{r}) = Y_-\mathbf{E}_-(\mathbf{r}). \tag{12}$$

Inserting $\mathbf{E}$ and $\mathbf{H}$ in terms of $\mathbf{E}_+$ and $\mathbf{E}_-$ in the Maxwell equations and requiring that there be no coupling between $\mathbf{E}_+$ and $\mathbf{E}_-$, leads to two quadratic dyadic equations for the admittances Y_+, Y_-:

$$Y_\mp^2\overline{\overline{\mu}} + Y_\mp(\overline{\overline{\xi}} + \overline{\overline{\zeta}}) + \overline{\overline{\epsilon}} = 0 \tag{13}$$

similar to (8). To be solvable, they require that the medium be of the self-dual type (2). In this notation, the admittances can be solved as $Y_\pm = \pm j/\eta_\pm$ with $\eta_\pm = \eta e^{\mp j\theta}$, and the decomposed fields become

$$\mathbf{E}_\pm = \frac{1}{2\cos\theta}(e^{\mp j\theta}\mathbf{E} \mp j\eta\mathbf{H}), \qquad \mathbf{H}_\pm = \frac{1}{2\cos\theta}(e^{\pm j\theta}\mathbf{H} \pm \frac{j}{\eta}\mathbf{E}). \tag{14}$$

This approach has led us to the decomposition in bi-isotropic media alternatively labeled as the Bohren decomposition [3], self-dual decomposition or wavefield decomposition [4]. It splits the Maxwell equations in two independent sets:

$$\nabla \times \mathbf{E}_\pm = -j\omega\overline{\overline{\mu}}_\pm \cdot \mathbf{H}_\pm - \mathbf{M}_\pm, \tag{15}$$

$$\nabla \times \mathbf{H}_\pm = j\omega\overline{\overline{\epsilon}}_\pm \cdot \mathbf{E}_\pm + \mathbf{J}_\pm, \tag{16}$$

when the decomposed electric and magnetic currents are defined as

$$\mathbf{J}_\pm = \frac{1}{2\cos\theta}(e^{\pm j\theta}\mathbf{J} \mp \frac{j}{\eta}\mathbf{M}), \qquad \mathbf{M}_\pm = \frac{1}{2\cos\theta}(e^{\mp j\theta}\mathbf{M} \pm j\eta\mathbf{J}), \tag{17}$$

and the equivalent permittivity and permeability dyadics as $\overline{\overline{\epsilon}}_\pm = \epsilon e^{\pm j\theta}\overline{\overline{\tau}}_\pm$, $\overline{\overline{\mu}}_\pm = \mu e^{\mp j\theta}\overline{\overline{\tau}}_\pm$, with the dyadics $\overline{\overline{\tau}}_\pm$ defined in (10). The decomposed fields and sources obey the relations,

$$\mathbf{E}_\pm = \mp j\eta_\pm\mathbf{H}_\pm, \qquad \mathbf{M}_\pm = \pm j\eta_\pm\mathbf{J}_\pm. \tag{18}$$

5. Green Dyadics

The fields generated by arbitrary electric and magnetic sources in a homogeneous linear bi-anisotropic medium can be expressed as integrals of four Green dyadics $\overline{\overline{G}}_{ee}(\mathbf{r})$, $\overline{\overline{G}}_{ee}(\mathbf{r})$, $\overline{\overline{G}}_{ee}(\mathbf{r})$ and $\overline{\overline{G}}_{mm}(\mathbf{r})$ satisfying

$$\left[\begin{pmatrix} 0 & \nabla \times \overline{\overline{I}} \\ -\nabla \times \overline{\overline{I}} & 0 \end{pmatrix} - j\omega\begin{pmatrix} \overline{\overline{\epsilon}} & \overline{\overline{\xi}} \\ \overline{\overline{\zeta}} & \overline{\overline{\mu}} \end{pmatrix}\right] \cdot \begin{pmatrix} \overline{\overline{G}}_{ee}(\mathbf{r}) & \overline{\overline{G}}_{em}(\mathbf{r}) \\ \overline{\overline{G}}_{me}(\mathbf{r}) & \overline{\overline{G}}_{mm}(\mathbf{r}) \end{pmatrix} = \begin{pmatrix} \overline{\overline{I}} & 0 \\ 0 & \overline{\overline{I}} \end{pmatrix}\delta(\mathbf{r}). \tag{19}$$

For the decomposed fields in a self-dual medium we only need to consider two Green dyadics $\overline{\overline{G}}_+(\mathbf{r})$ and $\overline{\overline{G}}_-(\mathbf{r})$, which can be defined in terms of fields due to the decomposed point sources. They satisfy

$$\overline{\overline{H}}_\pm(\nabla) \cdot \overline{\overline{G}}_\pm(\mathbf{r}) = (\nabla \times \overline{\overline{I}} \mp k\overline{\overline{\tau}}_\pm) \cdot \overline{\overline{G}}_\pm(\mathbf{r}) = \mp j\eta e^{\mp j\theta}\overline{\overline{I}}\delta(\mathbf{r}). \tag{20}$$

Relations between the different Green dyadics are

$$\overline{\overline{G}}_{ee}(\mathbf{r}) = \frac{1}{2\cos\theta}\left(e^{j\theta}\overline{\overline{G}}_+(\mathbf{r}) + e^{-j\theta}\overline{\overline{G}}_-(\mathbf{r})\right), \qquad \overline{\overline{G}}_{em}(\mathbf{r}) = \frac{1}{2j\eta\cos\theta}\left(\overline{\overline{G}}_+(\mathbf{r}) - \overline{\overline{G}}_-(\mathbf{r})\right), \tag{21}$$

$$\overline{\overline{G}}_{me}(\mathbf{r}) = \frac{-1}{2j\eta}\left(e^{j\theta}\overline{\overline{G}}_+(\mathbf{r}) - e^{-j\theta}\overline{\overline{G}}_-(\mathbf{r})\right), \qquad \overline{\overline{G}}_{mm}(\mathbf{r}) = \frac{1}{2\eta^2}\left(\overline{\overline{G}}_+(\mathbf{r}) + \overline{\overline{G}}_-(\mathbf{r})\right). \tag{22}$$

To find the solutions for (20), the dyadics $\overline{\overline{\tau}}_\pm$ are first expressed in terms of their symmetric and antisymmetric parts as $\overline{\overline{\tau}}_\pm = \overline{\overline{S}}_\pm + \mathbf{a}_\pm \times \overline{\overline{I}}$. The equation (20) then becomes

$$[\overline{\overline{L}}_\pm(\nabla) \mp k\mathbf{a}_\pm \times \overline{\overline{I}}] \cdot \overline{\overline{G}}_\pm(\mathbf{r}) = \mp j\eta e^{\mp j\theta}\overline{\overline{I}}\delta(\mathbf{r}), \qquad \overline{\overline{L}}_\pm(\nabla) = \nabla \times \overline{\overline{I}} \mp k\overline{\overline{S}}_\pm. \tag{23}$$

The Green dyadics $\overline{\overline{G}}_\pm(\mathbf{r})$ can now be expressed in terms of two scalar Green functions $g_\pm(\mathbf{r})$ satisfying a second-order equation (for details see [6])

$$\overline{\overline{G}}_\pm(\mathbf{r}) = \pm j\eta e^{\mp j\theta} e^{\pm k\mathbf{a}_\pm\cdot\mathbf{r}}\overline{\overline{L}}_\pm^{(2)T}(\nabla)g_\pm(\mathbf{r}), \quad \det\overline{\overline{L}}_\pm(\nabla)g_\pm(\mathbf{r}) = -\delta(\mathbf{r}). \tag{24}$$

The solutions for $g_\pm(\mathbf{r})$ are obtained through an affine transformation [2] as

$$g_\pm(\mathbf{r}) = \mp\frac{e^{-jkD_\pm}}{4\pi kD_\pm}, \qquad D_\pm(\mathbf{r}) = \sqrt{\det\overline{\overline{S}}_\pm}\sqrt{\overline{\overline{S}}_\pm^{-1} : \mathbf{rr}}. \tag{25}$$

Thus, the two Green dyadics obey the expressions

$$\overline{\overline{G}}_\pm(\mathbf{r}) = -j\eta e^{\mp j\theta} e^{\pm k\mathbf{a}_\pm\cdot\mathbf{r}}\overline{\overline{L}}_\pm^{(2)T}(\nabla)\left(\frac{e^{-jkD_\pm}}{4\pi kD_\pm}\right). \tag{26}$$

Finally, the four basic Green dyadics $\overline{\overline{G}}_{ee}\cdots\overline{\overline{G}}_{mm}$ are obtained by substituting (26) in (21) – (22). For example, the expression for the electric-electric Green dyadic becomes

$$\overline{\overline{G}}_{ee}(\mathbf{r}) = -\frac{j\eta e^{k\mathbf{a}_+\cdot\mathbf{r}}}{2\cos\theta}\overline{\overline{L}}_+^{(2)T}(\nabla)\left(\frac{e^{-jkD_+}}{4\pi kD_+}\right) - \frac{j\eta e^{-k\mathbf{a}_-\cdot\mathbf{r}}}{2\cos\theta}\overline{\overline{L}}_-^{(2)T}(\nabla)\left(\frac{e^{-jkD_-}}{4\pi kD_-}\right). \tag{27}$$

As a simple check we can consider the bi-isotropic special case by choosing $\overline{\overline{\alpha}} = \overline{\overline{I}}$ and $\overline{\overline{\kappa}}_r = \kappa_r\overline{\overline{I}}$. These imply $k\overline{\overline{\tau}}_\pm = k\overline{\overline{S}}_\pm = k_\pm\overline{\overline{I}}$, $kD_\pm = k_\pm r$ and $k_\pm = k(\cos\theta \pm \kappa_r)$, whence the well-known expression for $\overline{\overline{G}}_{ee}(\mathbf{r})$, as originally presented in [5], can be obtained.

6. Conclusion

The class of self-dual bi-anisotropic medium defined by two arbitrary dyadics (2) was shown to be definable in three different basic ways. It was shown that the Green dyadic corresponding to any bi-anisotropic medium in this class can be found in explicit analytic form by applying the decomposition method.

References

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