Static Image Principle for the Sphere in Bi-Isotropic Space

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Abstract

The classical inversion transformation in a sphere was originally introduced by Lord Kelvin to formulate the electrostatic image principle for the perfectly electrically conducting sphere. Here it shown that the inversion principle can be used as the basis of an extended electrostatic image principle for a perfectly magnetically conducting sphere and for a sphere with an impedance boundary condition. Any of these spheres may even reside in the most general linear isotropic medium, namely in the bi-isotropic medium, as will be shown.

1. Introduction

Inversion in a sphere (also called the Kelvin transformation) is one of the transformations that keep the Maxwell equations invariant in the static case. This conformal transformation involves one parameter—the radius a of the reference sphere. In spite of the basic limitation to statics, the inversion is applicable to time-harmonic problems, even at microwave frequencies. This requires that the region of interest be small enough in wavelengths, what is known as the quasi-static approximation. In this case the medium parameters are not the static ones but those taken at the frequency in question, in general with complex values.

The inversion method was introduced by William Thomson (later known as Lord Kelvin) in 1845 for solving static problems involving a perfectly electrically conducting (PEC) grounded sphere. The inversion method arises from the observation that one of the equi-potential surfaces (the one of zero potential) of two given point charges of different magnitude and opposite sign happens to be a sphere enclosing one of the charges, which can be seen through simple geometric reasoning. Because the sphere of zero potential can be covered by PEC material without changing the fields on either side of it, this immediately leads to an image principle for a point charge either inside or outside the PEC sphere.

In the present paper the inversion principle is first applied to the PEC sphere in an isotropic dielectric space for finding the classical Kelvin image priciple. Then, after some adjustment, it is applied to the perfectly magnetically conducting (PMC) sphere and to spherical surfaces with an impedance (mixed) boundary condition ('impedance spheres'). In these two cases the image of a point charge becomes a combination of point and line charges. The same approach is then used for an impedance sphere enclosed in bi-isotropic space. The result will include the images for the PEC and PMC spheres in bi-isotropic space and all types of spheres in isotropic space as special cases.

2. Kelvin's Inversion in Isotropic Dielectric Medium

The potential $\phi(\mathbf{r})$ from a source $\varrho(\mathbf{r})$ satisfies in homogeneous dielectric medium the Poisson equation

$$\nabla^2\phi(\mathbf{r}) = -\frac{\varrho(\mathbf{r})}{\epsilon}. \tag{1}$$

It can be shown [1, Appendix] that the same charge and potential functions also satisfy the equation

$$\nabla^2\left(\frac{a}{r}\phi(\frac{a^2}{r^2}\mathbf{r})\right) = -\frac{1}{\epsilon}\frac{a^5}{r^5}\varrho(\frac{a^2}{r^2}\mathbf{r}). \tag{2}$$

This is of the form (1) if we define the Kelvin-inverted potential and source functions as

$$\phi_{\mathrm{K}}(\mathbf{r}) = \frac{a}{r}\phi(\frac{a^2}{r^2}\mathbf{r}), \qquad \varrho_{\mathrm{K}}(\mathbf{r}) = \frac{a^5}{r^5}\varrho(\frac{a^2}{r^2}\mathbf{r}). \tag{3}$$

In the following discussion we assume the source function $\varrho(\mathbf{r})$ to be either completely outside or inside the spherical surface $r = a$.

2.1 PEC sphere

When $\mathbf{r}$ lies on the spherical surface $r = a$, we have from (3)

$$\phi_{\mathrm{K}}(\mathbf{r})|_{r=a} = \phi(\mathbf{r})|_{r=a}. \tag{4}$$

This means that the potential for the difference-charge $\varrho_{\mathrm{d}}(\mathbf{r}) = \varrho(\mathbf{r}) - \varrho_{\mathrm{K}}(\mathbf{r})$ vanishes on the sphere $r = a$:

$$\phi_{\mathrm{d}}(\mathbf{r})|_{r=a} = [\phi(\mathbf{r}) - \phi_{\mathrm{K}}(\mathbf{r})]_{r=a} = \left[\phi(\mathbf{r}) - \frac{a}{r}\phi(\frac{a^2}{r^2}\mathbf{r})\right]_{r=a} = 0. \tag{5}$$

Thus, the charge

$$\varrho_{\mathrm{PEC}}(\mathbf{r}) = -\varrho_{\mathrm{K}}(\mathbf{r}) \tag{6}$$

is the image of the charge $\varrho(\mathbf{r})$ in a grounded (zero-potential) PEC sphere $r = a$. This is the classical Kelvin image principle.

2.2 PMC sphere

The PMC boundary condition requires vanishing of the normal component of the electric field or the normal derivative of the electrostatic potential. Let us denote $\partial/\partial r$ by ∂_r for brevity. For the combined sum-charge

$$\varrho_{\mathrm{s}}(\mathbf{r}) = \varrho(\mathbf{r}) + \varrho_{\mathrm{K}}(\mathbf{r}) \tag{7}$$

the normal derivative of the corresponding potential on the sphere $r = a$ is (prime denotes the inverse radius $r' = a^2/r$)

$$\partial_r\phi_{\mathrm{s}}(\mathbf{r})|_{r=a} = \left[\partial_r\phi(\mathbf{r}) - \frac{a}{r^2}\phi(\mathbf{r}') - \frac{a^3}{r^3}\partial_{r'}\phi(\mathbf{r}')\right]_{r=r'=a} = -\left.\phi(\mathbf{r})\right|_{r=a} \tag{8}$$

and, as we see, the right-hand side does not vanish in general. However, it can be canceled by adding the source $(a\partial_r)^{-1}\varrho_{\mathrm{K}}(\mathbf{r})$. For the new sum-source $\varrho_{\mathrm{s}}(\mathbf{r}) = \varrho(\mathbf{r}) + (1 + 1/(a\partial_r))\varrho_{\mathrm{K}}(\mathbf{r})$ the normal derivative of the total potential on the sphere $r = a$ vanishes and the image of the charge function $\varrho(\mathbf{r})$ in the PMC sphere is thus

$$\varrho_{\mathrm{PMC}}(\mathbf{r}) = (1 + \frac{1}{a\partial_r})\varrho_{\mathrm{K}}(\mathbf{r}), \tag{9}$$

where the operator $1/\partial_r$ must be understood as integration along the r coordinate:

$$\varrho_{\mathrm{PMC}}(\mathbf{r}) = \varrho_{\mathrm{K}}(\mathbf{r}) + \frac{1}{a}\int_a^r \varrho_{\mathrm{K}}(\mathbf{r})\,dr. \tag{10}$$

Here the integration limit is chosen so that the image does not extend to the wrong side of the sphere $r = a$.

For example, for a point charge $\varrho(\mathbf{r}) = Q\delta(\mathbf{r} - \mathbf{r}_0)$ outside the sphere we have an image source consisting of a point charge $Q_{\mathrm{K}} = Qr_{\mathrm{K}0}/a$ at the Kelvin image point $\mathbf{r}_{\mathrm{K}0} = (a^2/r_0^2)\mathbf{r}_0$ plus a line charge of density $-Q_{\mathrm{K}}/a$ on the line connecting the Kelvin point to the center of the sphere.

2.3 Impedance sphere

A sphere with general linear boundary conditions is the impedance sphere on which the (total) potential satisfies

$$(\alpha + \beta\partial_r)\phi_{\mathrm{t}}(\mathbf{r})|_{r=a} = 0. \tag{11}$$

Here α and β are parameters which may have complex values in the quasi-static approximation. Inspired by the preceding theory, and after some algebraic manipulation, we find the image of the source $\varrho(\mathbf{r})$ in the operational form

$$\varrho_{\mathrm{imp}}(\mathbf{r}) = \left[1 + \frac{\beta - 2\alpha a}{a(\alpha + \beta\partial_r)}\right]\varrho_{\mathrm{K}}(\mathbf{r}). \tag{12}$$

The PEC and PMC cases above are obtained for $\beta \to 0$ and $\alpha \to 0$, respectively. The general impedance conditions of the form considered here are encountered, for example, in the radiation condition for time-harmonic potentials in the far zone where $\alpha/\beta = j\omega\sqrt{\mu_0\epsilon_0}$.

An inverse operator expression of the form

$$\frac{1}{\alpha + \beta\partial_r} f(r) = g(r) \tag{13}$$

can be expressed as the integral

$$g(r) = \frac{1}{\beta}\int_a^r e^{\alpha(q-r)/\beta} f(q)\,dq. \tag{14}$$

Again, the integration limit is chosen so that the image does not enter the wrong side of the sphere.

3. Kelvin's Inversion in Bi–Isotropic Medium

The most general linear and isotropic medium is characterized by a four-parameter constitutive equation of the form

$$\begin{pmatrix} \mathbf{D} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} \epsilon & \xi \\ \zeta & \mu \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}, \tag{15}$$

and the medium is called bi-isotropic. It is worth noting that because there is no coupling for static fields, the following analysis must be understood in a quasi-static sense.

The Poisson equations for the electric and magnetic scalar potential functions $\phi_{\mathrm{e}}(\mathbf{r})$, $\phi_{\mathrm{m}}(\mathbf{r})$ and the corresponding charge densities $\varrho_{\mathrm{e}}(\mathbf{r})$, $\varrho_{\mathrm{m}}(\mathbf{r})$ can be written in the matrix form

$$\nabla^2 \mathsf{f}(\mathbf{r}) = -\mathsf{M}^{-1}\mathsf{g}(\mathbf{r}), \tag{16}$$

where

$$\mathsf{f}(\mathbf{r}) = \begin{pmatrix} \phi_{\rm e}(\mathbf{r}) \\ \phi_{\rm m}(\mathbf{r}) \end{pmatrix}, \qquad \mathsf{M} = \begin{pmatrix} \epsilon & \xi \\ \zeta & \mu \end{pmatrix}, \qquad \mathsf{g}(\mathbf{r}) = \begin{pmatrix} \varrho_{\rm e}(\mathbf{r}) \\ \varrho_{\rm m}(\mathbf{r}) \end{pmatrix}. \tag{17}$$

The Kelvin-inverted quantities are now

$$\mathsf{f}_{\rm K}(\mathbf{r}) = \begin{pmatrix} \phi_{\rm eK}(\mathbf{r}) \\ \phi_{\rm mK}(\mathbf{r}) \end{pmatrix} = \frac{a}{r}\mathsf{f}(\frac{a^2}{r^2}\mathbf{r}), \qquad \mathsf{g}_{\rm K}(\mathbf{r}) = \begin{pmatrix} \varrho_{\rm eK}(\mathbf{r}) \\ \varrho_{\rm mK}(\mathbf{r}) \end{pmatrix} = \frac{a^5}{r^5}\mathsf{g}(\frac{a^2}{r^2}\mathbf{r}). \tag{18}$$

We shall study case of the impedance sphere directly, because it contains the previous PEC and PMC conditions as special cases. We combine two boundary operators in terms of two coefficients α and β as was done for the sphere in the dielectric medium, and get the generalized operator

$$\mathsf{B} = \begin{pmatrix} \alpha + \beta\epsilon\partial_r & \beta\xi\partial_r \\ \alpha\zeta\partial_r & \alpha\mu\partial_r + \beta \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} + \begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix} \mathsf{M}\partial_r. \tag{19}$$

Applying the previous line of thought we assume an image of the form

$$\mathsf{g}_{\rm imp}(\mathbf{r}) = \eta\mathsf{g}_{\rm K}(\mathbf{r}) + \mathsf{B}^{-1}\mathsf{C}\mathsf{g}_{\rm K}(\mathbf{r}). \tag{20}$$

When the boundary condition

$$\mathsf{B}\left[\mathsf{f}(\mathbf{r}) + \eta\mathsf{f}_{\rm K}(\mathbf{r})\right]_{r=a} + \mathsf{C}\mathsf{f}_{\rm K}(\mathbf{r})|_{r=a} = 0 \tag{21}$$

is expanded, η and C can be solved as

$$\eta = 1, \qquad \mathsf{C} = \begin{pmatrix} -2\alpha + \beta\epsilon/a & \beta\xi/a \\ \alpha\zeta/a & -2\beta + \alpha\mu/a \end{pmatrix} = -2\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} + \frac{1}{a}\begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix}\mathsf{M}. \tag{22}$$

Equation (20) is the most general image expression considered here and it gives all the other images as special cases. For example, setting $\alpha = 1$ and $\beta = 0$ leads to the PEC sphere case in a bi-isotropic medium, while setting $\xi = \zeta = 0$ leads to the impedance sphere in a dielectric and magnetic medium. The result coincides with (12) of the previous section when the difference in the definition of the parameter β is taken into account.

4. Conclusion

The inversion (Kelvin) transformation is known to produce the classical image principle for the PEC sphere even if in textbooks it is generally replaced by simpler geometric arguments allowing solving for the image of a point charge. However, the transformation is more applicable as it can produce images for the sphere with PMC and impedance boundary conditions. The latter result, given here, appears to be new. Also, as shown here, the method can be easily extended for generalizing a novel image principle to spheres in the bi-isotropic medium. Because the static approach can be used for time-harmonic problems as an approximation (quasi-static approximation), the image principle appears applicable whenever the radius of the sphere is small in terms of the wavelength.

References

[1] I. V. Lindell and J. J. Hänninen, "Static image principle for the sphere in isotropic or bi-isotropic space," *Radio Science*, vol. 35, no. 3, pp. 653–660, May–June 2000.