New Types of Orthonormal Electromagnetic Beams in Complex Media and Free Space

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Abstract

Vector plane–wave superpositions defined by a given set of orthonormal scalar functions on a two– or three–dimensional manifold are treated. We present a technique for composing orthonormal electromagnetic beams in complex media and free space. New families of exact time–harmonic solutions of the homogeneous Maxwell equations in linear media (isotropic, chiral, anisotropic, and bianisotropic) and free space, which describe orthonormal electromagnetic beams, are obtained.

1. Introduction

Properties of electromagnetic beams in composite and complex media (chiral, anisotropic, bianisotropic) are still investigated insufficiently. In some practical cases, for example, in characterizing complex media by free space techniques, the plane–wave approximation of incident beams proves to be inadequate. As in the case of conic refraction in biaxial crystals, in investigating possible physical phenomena caused by anomalous propagation in homogeneous [1, 2, 3, 4] and helicoidally nonhomogeneous [5, 6, 7] bianisotropic media, the plane–wave model of the incident beam is also inadequate. The most commonly used Hermite–Gaussian and Laguerre–Gaussian beams are mere the approximate solutions of Maxwell's equations in free space.

In Ref. [8], a novel technique for composing orthonormal electromagnetic beams and some other specific exact solutions of wave equations in linear media is suggested. By applying this technique, some new families of exact solutions of the homogeneous Maxwell equations for electromagnetic waves in isotropic media and free space are obtained in Refs. [8, 9, 10]. In this paper, we briefly outline the basic ideas of this technique and present its further applications.

2. Beam Manifold, Beam Base, and Beam State

Let (u_n) be a set of complex scalar functions $u_n : \mathcal{B}_u \to C^1$ on a real manifold $\mathcal{B}_u$, satisfying the orthogonality conditions

$$\langle u_m | u_n \rangle \equiv \int_{\mathcal{B}_u} u_m^*(b) u_n(b) d\mathcal{B} = \delta_{mn}, \tag{1}$$

where $d\mathcal{B}$ is the infinitesimal element of $\mathcal{B}_u$; u_m^* is the complex conjugate function to u_m, δ_{mn} is the Kronecker δ–function. Let us consider a plane–wave superposition (termed below the "beam" for brevity sake)

$$\begin{aligned} \mathbf{W}_n(\mathbf{x}) = & \int_{\mathcal{B}_u} e^{i\mathbf{x}\cdot\mathbf{K}(b)} u_n(b) \nu(b) \mathbf{W}(b) d\mathcal{B} \\ = & \int_{\mathcal{B}} e^{i\mathbf{x}\cdot\mathbf{K}(b)} u_n(b) \nu(b) \mathbf{W}(b) d\mathcal{B}, \end{aligned} \tag{2}$$

where $\mathcal{B} \subseteq \mathcal{B}_u$—beam manifold—is a subset of $\mathcal{B}_u$ with nonvanishing values of function $\mathbf{W}' = \nu(b)\mathbf{W}(b)$; $\mathbf{x}$ and $\mathbf{K}$ are the four–dimensional position and wave vectors. For electromagnetic waves, $\mathbf{W}$ can be any of the following quantities: the electric (magnetic) field vector $\mathbf{E}$ ($\mathbf{B}$), the four–dimensional field tensor F, or the six–dimensional vector $\mathrm{col}(\mathbf{E}, \mathbf{H})$. Functions $\mathbf{K} = \mathbf{K}(b)$ and $\mathbf{W} = \mathbf{W}(b)$ specify the set of plane harmonic waves involved in the beam (beam base), whereas a complex scalar function $\nu = \nu(b)$ specifies the beam state. The function $\mathbf{W}_n(\mathbf{x})$ is an exact solution of the linear field equations, provided that the integral in Eq. (2) exists.

The fields treated in this paper are composed of plane waves with factorized vector amplitudes $\mathbf{W}(b)\nu(b)u_n(b)d\mathcal{B}$, where each of the factors plays a specific part in determining the field structure. The normalized vector factor $\mathbf{W}(b)$ prescribes the polarization of the infinitesimal plane wave, whereas the scalar factors specify the intensity and the phase as follows. The function $\nu = \nu(b)$ is used mainly to obtain a set of orthonormal beams or to change the beam state [8]. In some special cases, it simply reduces to a normalizing constant factor. In the general case, the infinitesimal element $d\mathcal{B}$ depends on b, so that both u_n and $d\mathcal{B}$ act as weight functions defining the ratios in which plane waves with different propagation directions and polarization states are mixed in the beam $\mathbf{W}_n$. However, the distinctive features of $\mathbf{W}_n(\mathbf{x})$ as opposed to other beams are imprinted completely by the complex factor $u_n = u_n(b)$. This factorization is equally convenient both to set a unique wave pattern for each beam and to obtain the whole family of beams with a prescribed general property such as the orthonormality.

3. Time–Harmonic Orthonormal Beams

There are four key elements defining the properties of the presented beams: the set of complex scalar functions $u_n = u_n(b)$ on a real manifold $\mathcal{B}_u$, the beam manifold $\mathcal{B}$, the beam base functions $\mathbf{K} = \mathbf{K}(b)$ and $\mathbf{W} = \mathbf{W}(b)$, and the beam state function $\nu = \nu(b)$. By setting these elements in various ways, one can compose a multitude of normalized and orthonormal beams with very interesting properties [8, 9]. In particular, these elements can be set [8, 9] in such a way as to satisfy the orthonormality condition $s_{mn} \equiv \langle \mathbf{W}_m | Q | \mathbf{W}_n \rangle = N_Q \delta_{mn}$, where N_Q is some given scalar coefficient.

A time–harmonic beam of the form $\mathbf{W}_n$ [Eq. (2)] with a two–dimensional beam manifold $\mathcal{B}$, propagating in a homogeneous linear medium, can be written as

$$\mathbf{W}_n(\mathbf{r}, t) = e^{-i\omega t} \int_{\mathcal{B}} e^{i\mathbf{r}\cdot\mathbf{k}(b)} u_n(b)\nu(b)\mathbf{W}(b)d\mathcal{B}. \tag{3}$$

Let us introduce a scalar product

$$s_{mn} \equiv \langle \mathbf{W}_m | Q | \mathbf{W}_n \rangle = \int_{\sigma_0} \mathbf{W}_m^{\dagger}(\mathbf{r}, t) Q \mathbf{W}_n(\mathbf{r}, t) d\sigma_0, \tag{4}$$

where σ_0 is the plane with unit normal $\mathbf{q}$, passing through the point $\mathbf{r} = 0$, Q is some Hermitian operator, and $\mathbf{W}_m^{\dagger}(\mathbf{r}, t)$ is the Hermitian conjugate of $\mathbf{W}_m(\mathbf{r}, t)$. To compose orthonormal beams, it is convinient to set $\mathbf{W} = \mathrm{col}(\mathbf{E}, \mathbf{H})$, and

$$Q = \frac{c}{16\pi} \begin{pmatrix} 0 & -\mathbf{q}^{\times} \\ \mathbf{q}^{\times} & 0 \end{pmatrix}, \tag{5}$$

where $\mathbf{q}^{\times}$ is the antisymmetric tensor dual to $\mathbf{q}$ ($\mathbf{q}^{\times}\mathbf{E} = \mathbf{q} \times \mathbf{E}$). For a time–harmonic field, the normal component S_q of the time average Poynting vector $\mathbf{S}$ can be written as $S_q = \mathbf{q} \cdot \mathbf{S} = \mathbf{W}^{\dagger} Q \mathbf{W}$. Therefore, the condition $\langle \mathbf{W}_n | Q | \mathbf{W}_n \rangle = N_Q$ is in fact the normalization to the beam energy flux N_Q through the plane σ_0:

$$\langle \mathbf{W}_n | Q | \mathbf{W}_n \rangle = \int_{\sigma_0} S_q d\sigma_0 = N_Q. \tag{6}$$

We assume here that the tangential component $\mathbf{t}(b) = \mathbf{k}(b) - \mathbf{q}[\mathbf{q} \cdot \mathbf{k}(b)]$ of the wave vector $\mathbf{k}(b)$ is real for all $b \in \mathcal{B}$, and that the mapping $b \mapsto \mathbf{t}(b)$ is one-one (injective). It can be shown [8], that the beams $\mathbf{W}_n$ become orthonormal, if $\mathcal{B} = \mathcal{B}_u$, and the function $\nu(b)$ is given by

$$\nu(b) = \frac{1}{2\pi}\sqrt{\frac{N_Q J(b)}{g(b)\mathbf{W}^\dagger(b)Q\mathbf{W}(b)}}, \tag{7}$$

where $J(b) = D(t^j)/D(\xi^i)$ is the Jacobian determinant of the mapping $b \mapsto \mathbf{t}(b)$, calculated in terms of the local coordinate systems (ξ^i, $i = 1, 2$) on $\mathcal{B}$ and (t^j, $j = 1, 2$) on the $\mathbf{t}$-plane, preserving the orientation ($J(b) > 0$), and $d\mathcal{B} = g(b)d\xi^1 d\xi^2$. The expression under the square root in Eq. (7) has to be finite and positive almost everywhere, i.e., for all $b \in \mathcal{B}$ with the allowable exception of a set of measure zero in $\mathcal{B}$. This condition can be met with appropriately chosen beam manifold $\mathcal{B}$, operator Q, and normal $\mathbf{q}$. In some cases, when $\mathcal{B}$ is a proper subset of $\mathcal{B}_u$, i.e., $\mathcal{B} \neq \mathcal{B}_u$, the whole set of beams $\mathbf{W}_n$ [Eq. (3)] can not be orthonormalized; instead, its subset may be orthonormalized.

It is significant that the beam base functions $\mathbf{K} = \mathbf{K}(b)$ and $\mathbf{W} = \mathbf{W}(b)$, the beam state function $\nu = \nu(b)$, and the orthonormal functions $u_n = u_n(b)$ are defined on the same manifold $\mathcal{B}_u$. Therefore, the natural coordinates providing the coordinate representation of u_n, being used as integration variables, provide also the natural parametrization of the beams $\mathbf{W}_n(\mathbf{x})$ [Eq. (2)] or $\mathbf{W}_n(\mathbf{r}, t)$ [Eq. (3)]. In the case of beams defined by the spherical harmonics, treated in Refs. [8, 9, 10], $\mathcal{B}_u$ is a unit sphere. The polar angle θ and the azimuthal angle φ compose the natural coordinate system on it. These coordinates may coincide [8, 9] or be closely connected [10] with the spherical coordinates of the propagation direction, but this is by no means always the case. In particular, to compose the similar beams in a biaxially anisotropic medium, it is more advantageous to relate θ and φ with the biaxial coordinates [11]. These curvilinear coordinates yield the parametric representation of the wavevector surface, which makes possible to obviate the need for solving algebraic equations to describe this fourth order surface. They provide also a very convenient means for decription of field vectors of both eigenwaves and beams in biaxially anisotropic media.

If the beam $\mathbf{W}_n$ (3) consists of homogeneous eigenwaves, i.e., $\hat{\mathbf{k}}^*(b) = \hat{\mathbf{k}}(b)$ for all $b \in \mathcal{B}$, it may be of advantage to expand it into a series by using the Rayleigh formula

$$e^{i\mathbf{k}\cdot\mathbf{r}} = 4\pi \sum_{l=0}^{+\infty} i^l j_l(kr) \sum_{m=-l}^{l} Y_l^{m*}(\hat{\mathbf{k}}) Y_l^m(\hat{\mathbf{r}}) \ , \tag{8}$$

where $\hat{\mathbf{k}} = \mathbf{k}/k$, $\hat{\mathbf{r}} = \mathbf{r}/r$, and Y_l^m are the spherical harmonics. Substituting the expansion (8) into Eq. (3), we obtain

$$\mathbf{W}(\mathbf{r}, t) = e^{-i\omega t} \sum_{l=0}^{+\infty} i^l \sum_{m=-l}^{l} Y_l^m(\hat{\mathbf{r}}) \mathbf{W}_l^m(r) \ , \tag{9}$$

where

$$\mathbf{W}_l^m(r) = 4\pi \int_{\mathcal{B}} j_l(k(b)r) Y_l^{m*}(\hat{\mathbf{k}}(b)) \nu(b) u(b) \mathbf{W}(b) d\mathcal{B} \ . \tag{10}$$

Within the framework of this description, the beam is characterized by a set of radial vector functions $\mathbf{W}_l^m = \mathbf{W}_l^m(r)$. In an isotropic medium (achiral or chiral), these relations become

$$\mathbf{W}(\mathbf{r}, t) = e^{-i\omega t} \sum_{l=0}^{+\infty} i^l j_l(kr) \sum_{m=-l}^{l} Y_l^m(\hat{\mathbf{r}}) \mathbf{W}_l^m \ , \tag{11}$$

where the coordinate independent vector coefficients

$$\mathbf{W}_l^m = 4\pi \int_{\mathcal{B}} Y_l^{m*}(\hat{\mathbf{k}}(b))\nu(b)u(b)\mathbf{W}(b)d\mathcal{B} \tag{12}$$

completely characterize the beam.

4. Conclusion

New families of exact time–harmonic solutions of Maxwell's equations in homogeneous linear media (isotropic, chiral, anisotropic, and bianisotropic) and free space, which describe orthonormal electromagnetic beams, are obtained. Owing to the orthonormality conditions, these beams form convenient functional bases for more complex fields and provide a helpful technique for modelling the beams now in use and investigating their scattering and propagation in various media. The presented results form a basis for generalizing the wave–splitting technique developed in [12, 13] to the case of beams in plane–stratified media. They can be used for modelling of incident and scattered beams in investigating possible physical phenomena caused by anomalous propagation in homogeneous and helicoidally nonhomogeneous bianisotropic slabs.

In the poster presentation, we illustrate the obtained solutions by calculated spatial distributions of energy density, time average Poynting's vector, and local polarization parameters for a number of beams in complex media and free space.

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