Polarisability Analysis of Layered Bi-Anisotropic Ellipsoids

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Abstract

Using six-vector formalism, the polarisability problem of a layered bi-anisotropic ellipsoid is solved. The polarisability six-dyadic is explicitly calculated for the case of a two-component ellipsoid where the core is fully bi-anisotropic but the shell and the environment are bi-isotropic at most. It is reasonably straightforward to include more bi-isotropic layers on the ellipsoid. The limitation of the analysis is that all ellipsoidal boundaries for the composite structure have to be confocal.

1. Introduction

In this presentation we treat and analyse the calculation of the polarisability of a small scatterer. The polarisability is defined as the linear relation between the dipole moment and the uniform incident field that induces this moment. For anisotropic or non-spherical scatterers, the polarisability is generally a dyadic: $\mathbf{p} = \overline{\overline{\alpha}} \cdot \mathbf{E}$, where $\mathbf{p}$ is the dipole moment and $\mathbf{E}$ is the incident electric field. In the present paper we will concentrate on the case of bi-anisotropic scatterers in which case the polarisability is a matrix relation between the induced electric and magnetic dipole moments and the incident electric and magnetic fields. The concept of polarisability is important in the low-frequency applications [1]: the size of the scatterer has to be small in comparison of the wavelength of the operating field.

In the bi-anisotropic case we will use the six-vector formalism [2]. The electric and magnetic vector quantities are collected into a six-vector, and the relation between two six-vectors is a six-dyadic. Thus, for example, the polarisability is a six-dyadic (consisting of four ordinary three-dyadics, or 36 scalar parameters): $\mathsf{p} = \mathsf{A} \cdot \mathsf{e}$, where the field and dipole moment six-vectors are

$$\mathsf{p} = \begin{pmatrix} \mathbf{p_e} \\ \mathbf{p_m} \end{pmatrix} \quad \text{and} \quad \mathsf{e} = \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \tag{1}$$

In this presentation, the particular emphasis is on inhomogeneous scatterers; especially layered ellipsoidal scatterers. We will show how the polarisability six-matrix of a layered ellipsoidal scatterer can be calculated with the six-matrix formalism: the equations are shown for the case of core-plus-shell ellipsoid but the method works for any number of ellipsoidal layers. There is, however, one limitation: all the ellipsoidal boundaries in the structure have to be confocal. In other words, the foci of the various ellipsoids have to coincide. The results allow the layers (and the environment) to be bi-isotropic; in addition, the core of the layered ellipsoid can be arbitrarily bi-anisotropic.

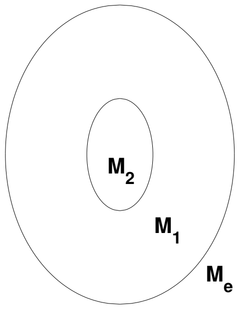

Figure 1: Layered ellipsoid with the core and shell

The results form a considerable generalisation of the previous studies where the corresponding polarisability problem for layered ellipsoidal dielectric scatterers [3] and for the two-layer chiral sphere [4] was solved.

2. Polarisability Six-Dyadic for the Core–Plus–Shell Ellipsoid

Consider the scatterer illustrated in Figure 1 where an ellipsoid is located in bi-isotropic environment. The ellipsoid consists of a bi-anisotropic core and a bi-isotropic layer (shell). It has to be remembered that the ellipsoidal boundaries have to be confocal, which roughly means that the core is less "rounded" than the whole scatterer. The material parameters in the problem are contained in the three material six-dyadics

$$\mathsf{M}_e = \begin{pmatrix} \epsilon_e \overline{\overline{I}} & \xi_e \overline{\overline{I}} \\ \zeta_e \overline{\overline{I}} & \mu_e \overline{\overline{I}} \end{pmatrix}, \qquad \mathsf{M}_1 = \begin{pmatrix} \epsilon_1 \overline{\overline{I}} & \xi_1 \overline{\overline{I}} \\ \zeta_1 \overline{\overline{I}} & \mu_1 \overline{\overline{I}} \end{pmatrix}, \qquad \mathsf{M}_2 = \begin{pmatrix} \overline{\overline{\epsilon}}_2 & \overline{\overline{\xi}}_2 \\ \overline{\overline{\zeta}}_2 & \overline{\overline{\mu}}_2 \end{pmatrix} \tag{2}$$

How to calculate the six-dyadic polarisability for this ellipsoid? Using the three-vector analysis as was done in [4] for the layered chiral sphere would lead to a large set of coupled vector equations from which the polarisability would be extremely tedious to solve. Here the six-vector analysis shows its power, because the analysis remains formally on the same level of complexity as in the three-vector case, and we can follow the steps in the layered dielectric studies.

The result is that the polarisability six-dyadic

$$\mathsf{A} = \begin{pmatrix} \overline{\overline{\alpha}}_{ee} & \overline{\overline{\alpha}}_{em} \\ \overline{\overline{\alpha}}_{me} & \overline{\overline{\alpha}}_{mm} \end{pmatrix} \tag{3}$$

of a layered bi-anisotropic ellipsoid can be expressed in a compact form:

$$\mathsf{A} = V \mathsf{M}_e \cdot \left\{ \mathsf{K} \cdot [(\mathsf{M}_1 - \mathsf{M}_e) \cdot \mathsf{L}_1 + \mathsf{M}_e] + w(\mathsf{M}_2 - \mathsf{M}_1) \cdot \mathsf{L}_1 \right\}^{-1} \cdot [\mathsf{K} \cdot (\mathsf{M}_1 - \mathsf{M}_e) + w(\mathsf{M}_2 - \mathsf{M}_1)] \tag{4}$$

with the following definitions:

$$\mathsf{K} = (\mathsf{M}_2 - \mathsf{M}_1) \cdot (\mathsf{L}_2 - w\mathsf{L}_1) \cdot \mathsf{M}_1^{-1} + \mathsf{I}$$

and the unit six-dyadic is defined obviously by

$$\mathsf{I} = \begin{pmatrix} \overline{\overline{I}} & 0 \\ 0 & \overline{\overline{I}} \end{pmatrix}$$

The depolarisation factors of the two ellipsoidal boundaries are contained in the six-dyadics

$$\mathsf{L}_1 = \begin{pmatrix} \overline{\overline{L}}_1 & 0 \\ 0 & \overline{\overline{L}}_1 \end{pmatrix} \quad \text{and} \quad \mathsf{L}_2 = \begin{pmatrix} \overline{\overline{L}}_2 & 0 \\ 0 & \overline{\overline{L}}_2 \end{pmatrix}$$

and w is fraction of the core from the total volume V of the particle.

The present analysis can be generalized to ellipsoids having more layers than the core and the shell, as in [3, 5].

3. Special Cases

3.1 Homogeneous sphere

It is important to check the result with special cases known earlierly. A trivial test could be to have a homogeneous isotropic sphere in isotropic environment, for which the scalar polarisability is

$$\alpha = 3V\epsilon_e \frac{\epsilon_i - \epsilon_e}{\epsilon_i + 2\epsilon_e} \tag{5}$$

It is quite easy to see that (4) passes this test.

3.2 Layered dielectric ellipsoid

The second special case is the layered dielectric (or magnetic) ellipsoid (see [3]): $\xi_e = \xi_1 = \zeta_e = \zeta_1 = 0$, $\overline{\overline{\xi}}_2 = \overline{\overline{\zeta}}_2 = 0$, $\overline{\overline{\epsilon}}_2 = \epsilon_2 \overline{\overline{I}}$, and $\overline{\overline{\mu}}_2 = \mu_2 \overline{\overline{I}}$. In this case the polarisability six-dyadic reduces to

$$\mathsf{A} = \begin{pmatrix} \overline{\overline{\alpha}}_{ee} & 0 \\ 0 & \overline{\overline{\alpha}}_{mm} \end{pmatrix}$$

where the electric polarisability is

$$\overline{\overline{\alpha}}_{ee} = V\epsilon_e \sum_{i=x,y,z} \frac{(\epsilon_1 - \epsilon_e)(\epsilon_1 + N_{2i}(\epsilon_2 - \epsilon_1)) + w(\epsilon_1 + N_{1i}(\epsilon_e - \epsilon_1))(\epsilon_2 - \epsilon_1)}{(\epsilon_e + N_{1i}(\epsilon_1 - \epsilon_e))(\epsilon_1 + N_{2i}(\epsilon_2 - \epsilon_1)) + wN_{1i}(1 - N_{1i})(\epsilon_1 - \epsilon_e)(\epsilon_2 - \epsilon_1)} \mathbf{u}_i\mathbf{u}_i \tag{6}$$

and the magnetic polarisability $\overline{\overline{\alpha}}_{mm}$ is, mutatis mutandis, the same as $\overline{\overline{\alpha}}_{ee}$. N_{1i} and N_{2i} are the depolarisation factors of the ellipsoids:

$$\overline{\overline{L}}_1 = N_{1x}\mathbf{u}_x\mathbf{u}_x + N_{1y}\mathbf{u}_y\mathbf{u}_y + N_{1z}\mathbf{u}_z\mathbf{u}_z \quad \text{and} \quad \overline{\overline{L}}_2 = N_{2x}\mathbf{u}_x\mathbf{u}_x + N_{2y}\mathbf{u}_y\mathbf{u}_y + N_{2z}\mathbf{u}_z\mathbf{u}_z.$$

3.3 Other special cases

Among the other possible special cases to test the result, the following ones can be considered: homogeneous chiral sphere and layered chiral sphere. The result (4) passes also these tests, but the resulting lengthy expressions are not shown here.

4. Maxwell Garnett Mixing Formula

With (4) we can build the Maxwell Garnett (Clausius–Mossotti) mixing formula for aligned layered ellipsoids:

$$\mathsf{M}_{eff} = \mathsf{M}_e + n\mathsf{M}_e \cdot \Big(\mathsf{M}_e - n\mathsf{A} \cdot \mathsf{L}_1\Big)^{-1} \cdot \mathsf{A} \tag{7}$$

where n is the number density of the layered ellipsoidal inclusions.

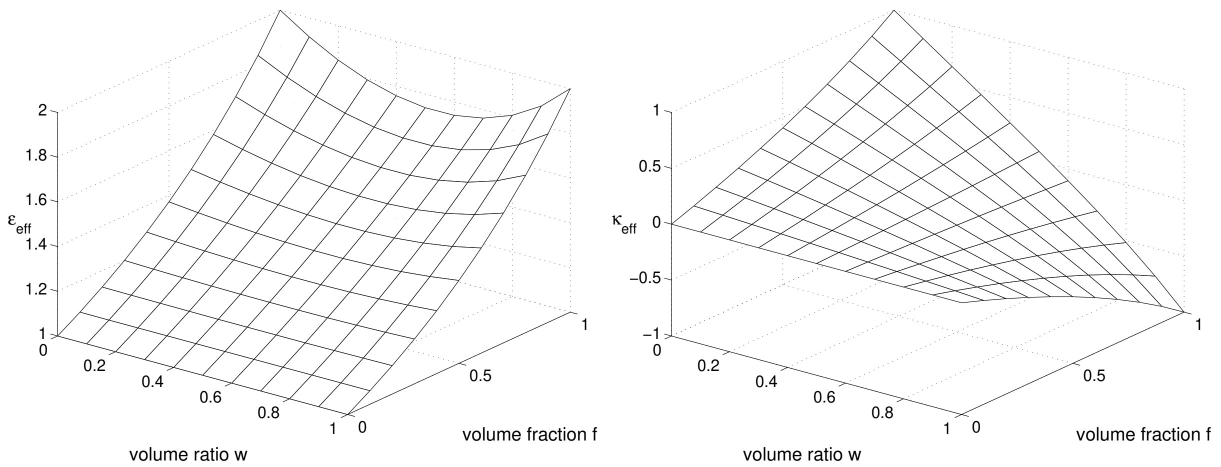

Figure 2: Effective relative permittivity (left) and chirality (right) of the mixture.

Numerical Example

Let us assume a mixture, which consists of layered chiral spheres in an isotropic background. The cores in the spheres are bi-isotropic: $\overline{\overline{\epsilon}}_2 = 2.0\epsilon_0\overline{\overline{I}}$, $\overline{\overline{\mu}}_2 = 1.5\mu_0\overline{\overline{I}}$, and $\kappa_2 = -1.0$, so that $\overline{\overline{\xi}}_2 = j\sqrt{\mu_0\epsilon_0}\overline{\overline{I}}$ and $\overline{\overline{\zeta}}_2 = -j\sqrt{\mu_0\epsilon_0}\overline{\overline{I}}$. The shells are also bi-isotropic: $\overline{\overline{\epsilon}}_1 = 2.0\epsilon_0\overline{\overline{I}}$, $\overline{\overline{\mu}}_1 = 1.5\mu_0\overline{\overline{I}}$, but $\kappa_1 = +1.0$. In other words: the materials in the layered spheres are almost the same, but the cores are right-handed and the shells are left-handed. For the background medium: $\epsilon_e = \epsilon_0$ and $\mu_e = \mu_0$.

It is a quite simple task to write a short Matlab-code and use (4) and (7), to calculate the effective material properties of the given mixture. In Figure 2 is shown the effective relative permittivity and chirality of the mixture. The volume fraction f is defined as $f = nV$, where n is the number density of the inclusions in the mixture and V is the total volume of an inclusion.

Figure 2 shows that although both components of the spheres have the same electric permittivity, the mixture permittivity ϵ_{eff} is not the same in the case $f = 1$ (no background, everything just inclusions). The strange effect on permittivity is caused by the magnetoelectric coupling.

References

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