Complex Media in Complex Fields: A Statistical Approach

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Abstract

The use of statistical techniques to characterize composite materials systems and their use in complex electromagnetic environments is discussed. Examples of the calculation of uncertainties and distribution functions of wave statistics are given.

1 Introduction

Heterogeneous media which consist of discrete inclusions of various kinds and configured in certain arrangements inside a host medium can be considered as complex electromagnetic (EM) material systems. A rigorous deterministic analytical or computational approach to their analysis is of almost forbidding complexity. Hence one usually takes recourse to simplified effective medium theories. Typically, only mean values (first-order moments) are considered, but this tends to be a significant oversimplification when correlating theoretically predicted characteristics of a composite with measured results for a realistic sample. In particular, variability between different realizations of samples proves to be a dominant factor in the uncertainty budget for the macroscopic constitutive parameters. The resulting uncertainties are furthermore important in assessing the accuracy of the measurement itself, in order to decide on the significance of certain measured effects (e.g. nonreciprocity or chirality), or on the relevant truncation point in any series expansion for their characterization in the long-wavelength regime. In crystal physics, the number of scattering centers is usually very large so as to warrant a negligible level of uncertainty. However for synthetic composites containing a relatively small number of inclusions, this is generally not the case. The uncertainty associated with sample realizations calls for new or improved methods for their characterization. Statistical characterization proves to be an especially powerful method, because the uncertainties *decrease* with increasing degree of complexity of the material system. The results find application, for example, in the characterization of adaptive material systems in active or passive mode of operation [1].

Secondly, a growing trend exists towards the use of complex EM environments as alternative measurement techniques, in order to characterize materials in their operational environment more realistically and accurately. Here, realistic excitation and illumination conditions are being generated, as opposed to the idealized case of single plane-wave illumination. The incident wave must then be considered as quasi-statistical, for the measured effect is due to an ensemble average of a multitude of different directions of incidence or polarization. The same is usually true for the internal field inside the medium even if the externally incident wave is deterministic. One such complex EM environment simulator is the *NPL* stadium reverberation chamber, in which an ensemble of illuminations is being generated for the EMC testing of equipment or characterization of EM materials or systems [2, 3]. The most general case is of course the one of complex media subjected to complex fields.

2 Statistics of waves and media

2.1 Moments

In the general case of complex media subject to complex waves, both the excitation and the constitutive parameters exhibit statistical fluctuations. Sample variability can be taken into account by incorporating a continuous or discrete *realization parameter* τ. For example, in the Lorenz–Lorentz formula the number of inclusions and their dipolarizabilities α thus become random variables. At a given frequency ω:

$$\epsilon(\tau;\omega) = \epsilon_0 + \frac{N(\tau)\ \alpha(\tau;\omega)}{1 - N(\tau)\ \alpha(\tau;\omega)/(3\epsilon_0)} \tag{1}$$

The average of the macroscopic electric polarization $P_e = N\alpha/[1 - N\alpha/(3\epsilon_o)]E$ then satisfies:

$$\langle P_e \rangle = \frac{\langle N\alpha \rangle E + N\alpha \langle E \rangle + \frac{\langle N\alpha \rangle}{3\epsilon_0} P_e}{1 - \frac{N\alpha}{3\epsilon_o}} \tag{2}$$

where $\langle \cdot \rangle$ signifies ensemble averaging. Its variance then follows from $\sigma^2_{P_e} = \langle P_e^2 \rangle - \langle P_e \rangle^2$, which also takes the deviation of the spatial distribution of the inclusions from a pure random distribution into account. With $\underline{D} = \epsilon_o \underline{E} + \underline{P}_e$ we can calculate σ^2_D. Its expression for general bianisotropic media has been obtained but is cumbersome. For the case of an anisotropic dielectric:

$$\begin{aligned} \sigma^2_D &= \langle \underline{D}^\dagger \cdot \underline{D} \rangle - \langle \underline{D}^\dagger \rangle \cdot \langle \underline{D} \rangle \\ &= \left(\underline{E}^\dagger \cdot \langle \underline{\underline{\epsilon}}^\dagger \rangle \cdot \langle \underline{\underline{\epsilon}} \rangle \cdot \underline{E} \right) + \left(\langle \underline{E}^\dagger \rangle \cdot \underline{\underline{\epsilon}}^\dagger \cdot \underline{\underline{\epsilon}} \cdot \langle \underline{E} \rangle \right) + 2 \left(\underline{E}^\dagger \cdot \langle \underline{\underline{\epsilon}}^\dagger \rangle \cdot \underline{\underline{\epsilon}} \cdot \langle \underline{E} \rangle \right) - \langle \underline{E}^\dagger \cdot \underline{\underline{\epsilon}}^\dagger \rangle \cdot \langle \underline{\underline{\epsilon}} \cdot \underline{E} \rangle \end{aligned} \tag{3}$$

A similar relation exist for σ^2_B. For an isotropic medium, (3) can be simplified to:

$$\sigma^2_D \ = \ \langle D^2 \rangle - \langle D \rangle^2 \ = \ \sigma^2_\epsilon E^2 + \epsilon^2 \sigma^2_E - 2\epsilon E \langle \epsilon \rangle \langle E \rangle \tag{4}$$

where σ_ϵ represents the standard deviation resulting from differences in material processing conditions, as well as differences or uncertainties in the number, spatial distribution, size, shape, etc. of the inclusions.

2.2 Distributions

A full characterization of the wave or medium ensemble requires knowledge of all higher-order statistical moments or, equivalently, the associated probability density function (pdf). Provided the pdf of the constitutive effective medium parameters is known a priori (which is often the case owing to the central limit theorem), wave statistics such as field correlation, wave number, wave impedance, power spectral density etc. can be derived. If the medium can be considered as 'random', which presumes that the number of inclusions is sufficiently large, the ensemble permittivity and permeability exhibit an approximately Gauss normal distribution, on physical grounds. For the random wavenumber K or random refractive index N, the resulting distribution is then obtained upon the subsequent variate transformations $x = \mu\epsilon$ and $n = \sqrt{x}$ as:

$$f_N(n) = \frac{2n}{\pi\sigma_\mu\sigma_\epsilon} K_0 \left[\frac{n^2}{\sigma_\mu\sigma_\epsilon} \right] \tag{5}$$

where $K_0(\cdot)$ is the modified Bessel functions of the 2nd kind of order zero. For the wave random impedance Z, the transformations $y = \mu/\epsilon$, $z = \sqrt{y}$ yield:

$$f_Z(z) = \frac{\sqrt{z}}{\pi \left[(\sigma_\mu/\sigma_\epsilon) z^2 + (\sigma_\epsilon/\sigma_\mu) \right]} \tag{6}$$

For the mean-normalized, statistically isotropic, homogeneous and unpolarized field, whose three complex components exhibit a circular Gauss normal distribution, the real (r) and imaginary (i) parts of $\underline{D}$ and $\underline{B}$ are distributed as:

$$f_{D^{r,i}}(d^{r,i}) = (\pi\sigma_\epsilon\sigma_E)^{-1} K_0 \left[\frac{d^{r,i}}{\sigma_\epsilon\sigma_E} \right] \tag{7}$$

The random power density S associated with this isotropic field satisfies, for deterministic constitutive parameters, a χ^2_{2p} pdf:

$$f_S(s) = \frac{s^{p-1} \ \exp(-s/2)}{2^p \ (p-1)!} \tag{8}$$

with $p = 1$ for a Cartesian component and $p = 3$ for the total rms power. For the field magnitude, a χ_{2p} pdf applies:

$$f_{|E_\alpha|}(|e_\alpha|) = \frac{|e_\alpha|^{2p-1} \ \exp(-|e_\alpha|^2/2)}{2^{p-1} \ (p-1)!} \tag{9}$$

Other wave statistics are obtained along similar lines, starting from basic normal random components with a Gauss normal distribution.

2.3 Spatial dispersion

Finite-size effects manifest themselves not only in the size of the material sample, but also in that of the inclusions themselves. For random fields, the effect of the latter on the effective properties can be taken into account by applying theoretical results for general local averaging [2]. The analysis then shows that finite size effects give rise to an decrease in the uncertainty of the constitutive parameters, but to an increase in the *perceived* (i.e. measured) randomness of the medium, as measured by its normalized spectral bandwidth. Typically, for sufficiently small size of the inclusions, the uncertainty will vary according the square of the characteristic length of the inclusion, or with the square of the averaging distance for the statistical field.

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References

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