Effective Constitutive Tensors of Bianisotropic Multilayered Mediums

A. N. Borzdov

Department of Theoretical Physics, Belarusian State University
Fr. Skaryny avenue 4, 220050 Minsk, Belarus
Fax: + 375 – 172 – 06 60 01; E-mail: alexbor@phys.bsu.unibel.by

Abstract

Operator formalism is employed for introduction of effective constitutive tensors of bianisotropic multilayered periodic strucures in a wide wave band. It is based on the approximate calculation of the characteristic matrix of the unit cell of the system with the help of Campbell–Hausdorff series [1]. In this paper we show that effective constitutive tensors can be correctly introdused for a greate variety of mediums.

1. Introduction

One of the most fruitful approaches in the theory of propagation of waves in multilayered media is the operator formalism based on the use of intrinsic representation of vectors and operators (see [2], [3], for example). It enables one to find compact coordinate–free formulae for very complex systems and to avoid the cumbersome calculations required by usual components techniques and caused mainly by the necessity of adopting a different coordinate system for each different layer.

In this paper we use this formalism to find the effective constitutive tensors of the medium formed by a periodic set of plane bianisotropic layers with different thicknesses l_n ($n = 1, 2, \dots N$, where N is the number of the layers which constitute the unit cell).

In what follows we assume the constitutive relations

$$\mathbf{D}_n = \varepsilon_n \mathbf{E}_n + \alpha_n \mathbf{H}_n, \ \mathbf{B}_n = \beta_n \mathbf{E}_n + \mu_n \mathbf{H}_n, \tag{1}$$

for layer's materials, where ε_n, μ_n and α_n, β_n are the dielectric permittivity, the magnetic permeability tensors and the pseudotensors of gyrotropy, respectively. These equations can be also written in matrix form, as

$$\begin{pmatrix} \mathbf{D}_n \\ \mathbf{B}_n \end{pmatrix} = \mathcal{R}_n \begin{pmatrix} \mathbf{E}_n \\ \mathbf{H}_n \end{pmatrix}, \quad \mathcal{R}_n = \begin{pmatrix} \varepsilon_n & \alpha_n \\ \beta_n & \mu_n \end{pmatrix}. \tag{2}$$

If $\varepsilon_n,\ \mu_n,\ \alpha_n,\ \beta_n$ ($n = 1, 2, \dots N$) are complex nonsymmetric tensors, then equations (1) or (2) describe an absorbing anisotropic and gyrotropic medium, subject to the influence of external electric and magnetic fields and elastic deformations.

The main purpose of this paper is to introduce the medium's effective material tensors, feasible for the use in the wide wave band.

2. Effective Constitutive Tensors in Wide Wave Band

To treat this problem we use here an approach to introduce the effective material parameters, proposed in [4]. It is based on the approximate calculation of the characteristic matrix P of the unit cell of the system with the help of Campbell–Hausdorff series [1].

The characteristic matrix $P = \exp(ik_0 lM)$ of a layer with thickness l relates the six-vectors $(\mathbf{E}, \mathbf{H})^T$ at the layer boundaries ($k_0 = \omega/c$). The matrix M can be written in the form [5]

$$M = \mathcal{T}_q Q_\times (\mathcal{R} - \mathcal{B}_\times) \mathcal{T}_q, \tag{3}$$

$$Q_\times = \begin{pmatrix} \mathbf{0} & -\mathbf{q}^\times \\ \mathbf{q}^\times & \mathbf{0} \end{pmatrix}, \ \mathcal{B}_\times = \begin{pmatrix} \mathbf{0} & -\mathbf{b}^\times \\ \mathbf{b}^\times & \mathbf{0} \end{pmatrix}, \tag{4}$$

$$\mathcal{T}_q = \mathcal{E} - \mathcal{N}_q, \ \mathcal{E} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathcal{N}_q = \mathcal{R}_q^{-1} Q (\mathcal{R} - \mathcal{B}_\times), \ Q = \begin{pmatrix} \mathbf{q}\otimes\mathbf{q} & \mathbf{0} \\ \mathbf{0} & \mathbf{q}\otimes\mathbf{q} \end{pmatrix}, \tag{5}$$

where operator $\mathcal{R}_q^{-1}$ is the inverse to the operator

$$\mathcal{R}_q = \begin{pmatrix} \varepsilon_q \mathbf{1} & \alpha_q \mathbf{1} \\ \beta_q \mathbf{1} & \mu_q \mathbf{1} \end{pmatrix}, \tag{6}$$

$$\varepsilon_q = \mathbf{q}\varepsilon\mathbf{q}, \mu_q = \mathbf{q}\mu\mathbf{q}, \alpha_q = \mathbf{q}\alpha\mathbf{q}, \beta_q = \mathbf{q}\beta\mathbf{q},$$

$\mathbf{q}^\times$ is the antisymmetric dyadic dual to the unit normal of the boundaries $\mathbf{q}$, $\mathbf{b}$ is the tangential component of refraction vector $\mathbf{m}$ [2],[3], $\mathbf{q}\otimes\mathbf{q}$ is the dyad and $\mathbf{1}$ is the unit dyadic.

In the case of a medium formed by two alternate layers with different thicknesses l_n and different sets of tensor constants $\varepsilon_n, \mu_n, \alpha_n, \beta_n$ $(n = 1, 2)$, the characteristic matrix of the unit cell has the form

$$P = \exp(ik_0 LM) = \exp(ik_0 l_2 M_2) \exp(ik_0 l_1 M_1), \tag{7}$$

where $L = l_1 + l_2$ is the system period and M is some matrix. This matrix can be expressed in terms of the layers parameters with the help of Campbell–Hausdorff series as follows

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = f_1 M_1 + f_2 M_2 + i f_1 \frac{\pi l_2}{\lambda} [M_2, M_1] - \cdots \tag{8}$$

where $f_n = l_n/L$ is the relative thickness of the n-th layer, $\sum_{n=1}^N f_n = 1$, $[M_2, M_1] = M_2 M_1 - M_1 M_2$. If $\pi l_n/\lambda \ll 1$, then the series (8) quickly converges and one can drop the remainder of it after the k–th term. Here for the sake of simplicity we have limited our consideration by three terms of series. The first and the second terms correspond to the long wavelength limit.

The tensor parameters A, B, C, D can be directly used for the analysis of the wave propagation in the considered system, for example for finding the reflection and transmission tensors [3].

For the sake of simplicity we assume $\mathbf{q}$ to be the left and the right eigenvector of each of the tensors $\varepsilon_n, \mu_n, \alpha_n, \beta_n$ $(\varepsilon_n \mathbf{q} = \mathbf{q}\varepsilon_n \equiv \varepsilon_{nq}\mathbf{q}, \cdots)$. Besides, we shall use in (7)–(8) the matrices M_I, M_{1I}, M_{2I}, describing the transformation of the tangential components of the field vectors, instead of M, M_1, M_2, describing the transformation of the full six-vectors. In the majority of applications this would be ample. Under above–mentioned conditions:

$$M_{nI} = Q_\times \left[\mathcal{R}_n - \frac{1}{\delta_n} \mathcal{R}_{nq}^T \begin{pmatrix} \mathbf{a}\otimes\mathbf{a} & \mathbf{0} \\ \mathbf{0} & \mathbf{a}\otimes\mathbf{a} \end{pmatrix} \right] \mathcal{T}, \tag{9}$$

$$\mathcal{T} = \left(\begin{array}{cc} I & 0 \\ 0 & I \end{array} \right), \; I = \mathbf{1} - \mathbf{q} \otimes \mathbf{q}, \tag{10}$$

where $\mathcal{R}_{nq}^T$ is the transposed operator, $\mathbf{a} = \mathbf{b}\mathbf{q}^{\times}$,

$$[M_{2I}, M_{1I}] = Q_{\times} \left[\mathcal{R}_{21} - \Delta(\mathbf{a}) \right] \mathcal{T}, \tag{11}$$

$$\mathcal{R}_{21} = \mathcal{R}_2 Q_{\times} \mathcal{R}_1 - \mathcal{R}_1 Q_{\times} \mathcal{R}_2, \tag{12}$$

$$\Delta(\mathbf{a}) = \Delta_L \left(\begin{array}{cc} \mathbf{a} \otimes \mathbf{a} & \mathbf{0} \\ \mathbf{0} & \mathbf{a} \otimes \mathbf{a} \end{array} \right) + \left(\begin{array}{cc} \mathbf{a} \otimes \mathbf{a} & \mathbf{0} \\ \mathbf{0} & \mathbf{a} \otimes \mathbf{a} \end{array} \right) \Delta_R, \tag{13}$$

where

$$\Delta_L = \left[\mathcal{R}_{2I} Q_{\times} \frac{\mathcal{R}_{1q}^T}{\delta_1} - \mathcal{R}_{1I} Q_{\times} \frac{\mathcal{R}_{2q}^T}{\delta_2} \right], \; \Delta_R = \left[\frac{\mathcal{R}_{2q}^T}{\delta_2} Q_{\times} \mathcal{R}_{1I} - \frac{\mathcal{R}_{1q}^T}{\delta_1} Q_{\times} \mathcal{R}_{2I} \right], \tag{14}$$

and $\mathcal{R}_{nI} = \mathcal{T}\mathcal{R}_n\mathcal{T}, n = 1, 2$.

For the correct introduction of permittivity, permeability and gyrotropy tensors of the medium equivalent in the considered approximation to the system at hand, the matrix M_I is bound to depend on the effective tensors, angle and plane of incidence in the same way as in the homogeneous medium. From (8)–(13) it follows that these conditions are met if $\Delta(\mathbf{a}) = 0$. In this case we can find the effective material tensors in the same way as in [6]

$$\mathcal{R} = \left(\begin{array}{cc} \varepsilon & \alpha \\ \beta & \mu \end{array} \right) = \mathcal{T} \left(f_1 \mathcal{R}_1 + f_2 \mathcal{R}_2 + i f_1 \frac{\pi l_2}{\lambda} \mathcal{R}_{21} \right) \mathcal{T} + \mathcal{R}_q Q, \tag{15}$$

with

$$\mathcal{R}_q = \left(f_1 \mathcal{R}_{1q}^{-1} + f_2 \mathcal{R}_{2q}^{-1} \right)^{-1}. \tag{16}$$

For example, $\Delta(\mathbf{a}) = 0$ if $\Delta_L = \Delta_R = 0$. That takes place provided [6]

$$\mathcal{R}_{2I} \mathcal{R}_{2q} = \mathcal{R}_{1I} \mathcal{R}_{1q}, \; \mathcal{R}_{2q} \mathcal{R}_{2I} = \mathcal{R}_{1q} \mathcal{R}_{1I}, \tag{17}$$

Note, that both relations (17) are equivalent if $\alpha_i = \beta_i = 0$. Even so it is rigid enough requirement, signifying that tensors ε_{1I} and ε_{2I}, μ_{1I} and μ_{2I} bound to have the same eigenvectors.

Of course, above mentioned conditions need not be met exactly having regard to approximate nature of method. We can find valuations $\|\Delta_L\|$ and $\|\mathcal{R}_{21}\|$ instead. It should be more than sufficient if $\|\Delta_L\|, \|\Delta_R\|$ far less than $\|\mathcal{R}_{21}\|$. In what follows we shall use Euclidean valuation

$$\|X\| = \left[(X X^{\dagger})_t \right]^{\frac{1}{2}}, \tag{18}$$

where $X^{\dagger}$ is Hermitian transposed tensor and $(XX^{\dagger})_t$ is a trace of tensor $XX^{\dagger}$.

By way of example let us consider nongyrotropic, nonmagnetic layers with real symmetric tensors ε_i. In this instance

$$\|\Delta_L\|^2 = \left(\frac{\varepsilon_{2+}}{\varepsilon_{1q}} - \frac{\varepsilon_{1+}}{\varepsilon_{2q}} \right)^2 + \left(\frac{\varepsilon_{2-}}{\varepsilon_{1q}} - \frac{\varepsilon_{1-}}{\varepsilon_{2q}} \right)^2 + 2 \sin \phi^2 \left(\frac{\varepsilon_{1+}}{\varepsilon_{1q}} - \frac{\varepsilon_{1-}}{\varepsilon_{1q}} \right) \left(\frac{\varepsilon_{2+}}{\varepsilon_{2q}} - \frac{\varepsilon_{2-}}{\varepsilon_{2q}} \right), \tag{19}$$

$$\|\mathcal{R}_{21}\|^2 = 2 \left[(\varepsilon_{2+} - \varepsilon_{1+})^2 + (\varepsilon_{2-} - \varepsilon_{1-})^2 + 2 \sin \phi^2 (\varepsilon_{1+} - \varepsilon_{1-}) (\varepsilon_{2+} - \varepsilon_{2-}) \right], \tag{20}$$

where ϕ is the angle between the preferred axes of crystals, $\varepsilon_{i+}, \varepsilon_{i-} (i = 1, 2)$ are the eigenvalues of tensors $\varepsilon_{iI} = I \varepsilon_i I$. Formulae for magnetic crystals are similarly to (19),(20). Since usually

$\|\alpha_i\|, \|\beta_i\| \ll \|\varepsilon_i\|, \|\mu_i\|$, corresponding formulas are valid also for analysis of the gyrotropic layers.

Inasmuch $\mathbf{a^2} \sim \varepsilon_{iq}$, from (19), (20), (13) it follows that for identical layers ($\varepsilon_{1+} = \varepsilon_{2+}, \varepsilon_{1-} = \varepsilon_{2-}$) values $\|\Delta(\mathbf{a})\|$ and $\|\mathcal{R}_{21}\|$ are approximately equal. Situation changes in the case of different parameters of the layers. In this case it is possible to choose parameters of the crystals providing the fulfillment of condition

$$\|\Delta(\mathbf{a})\| \ll \|\mathcal{R}_{21}\|. \tag{21}$$

Relation (21) can be provided much easier for structures comprising more then three layers in the unit cell. Then

$$\begin{aligned} M &= f_1M_1 + f_2M_2 + f_3M_3 + \\ &\frac{i\pi}{\lambda}\{f_3l_2[M_3, M_2] + f_3l_1[M_3, M_1] + f_1l_2[M_2, M_1]\} + \ldots, \end{aligned} \tag{22}$$

and instead of $\|\Delta(\mathbf{a})\|$, $\|\mathcal{R}_{21}\|$ (12)–(14) we shall have more complicated but more versatile expressions, which can be changed by varying thicknesses of the layers. It should be noted that sometimes the "wide wave band approximation" reduces to the long wavelength approximation. In particular, if $\mathcal{R}_1 = \mathcal{R}_3,\ l_1 = l_2 = l_3$, then $[M_3, M_1] = 0,\ [M_3, M_2] = -[M_2, M_1]$ and structure do not possess properties of form gyrotropy [6].

Formulas (15), (9)-(12) are valid for arbitrary material parameters of the layers for a small enough angles of incidence and, of course, in the case of normal incidence ($\mathbf{a} = \mathbf{0}$).

3. Conclusion

It is shown that effective constitutive tensors, not depending on the angle and plane of incidence (true constitutive tensors), can be introduced in the wide wave band for the great variety of anisotropic periodic mediums. Formulas, which enable one to determine whether it is possible to introduce true constitutive tensors for the structure at hand or not, are obtained. In the frame of considered approximation it is hardly probable to introduce correctly constitutive tensors for an arbitrary parameters of the layers. From the other hand, at normal incidence there are no limitations on the parameters of the layers and obtained effective tensors may be very useful for synthesis of new materials.

References

[1] J. P. Serre, *Lie Algebras Lie Groups.* New York–Amsterdam: Benjamin, 1965.

[2] F. I. Fedorov, *Theory of Gyrotropy.* Minsk: Nauka i Tekhnika, 1976.

[3] L. M. Barkovskii, G . N. Borzdov and A. V . Lavrinenko, "Fresnel's reflection and transmission operators for stratified gyroanisotropic media", *J. Phys. A: Math.Gen.*, vol. 20, pp. 1095–1106, 1987.

[4] A. N. Borzdov, "Effective properties of plane stratified bianisotropic systems," in *Proc. Bianisotropics'97*, University of Glasgow, Great Britain, June 1997, pp. 145–148.

[5] G. N. Borzdov, "Evolution operators of electromagnetic waves in crystals," *Sov. Phys. Cryst.*, vol. 35, no. 3, pp. 535–551, May–June 1990.

[6] A. N. Borzdov, "Operator formalism in the theory of effective parameters of plane stratified bianisotropic structures," *Electromagnetics*, vol. 19, no. 6, pp. 501–512, November–December 1999.